6.854J / 18.415J Advanced Algorithms
Fall 2008

1 Introduction

In this lecture we shall study Klein's cycle cancelling algorithm for finding the circulation of minimum cost in greater detail. We will pay particular attention to the choice of cycle to cancel and we will rigorously prove two bounds on the number of iterations required, the first of which depends on the magnitude of the cost and is valid only for integer-valued costs, and the second of which is strongly polynomial and works even for irrational costs.

Recall from last time that for a given circulation f, the following are equivalent:

i. f is of minimum cost

ii. There is no negative cost cycle in the residual graph G_f

iii. There exist potentials $p : V \to \mathbb{R}$ such that the reduced costs

$$c_p(v, w) = c(v, w) + p(v) - p(w) \geq 0$$

for all $(v, w) \in E_f$, where $E_f = \{e \ : \ u_f(e) > 0\}$.

2 Klein's cycle cancelling algorithm

Algorithm 1 KLEINS-CYCLE-CANCEL(G_f)

Let f be any circulation (e.g., f = 0)
while there exists a negative cost cycle $\Gamma \in G_f$ **do**
  Push $\epsilon(f) = \min\limits_{(v,w)\in\Gamma} u_f(v, w)$ along Γ
end while

It is important to note that the Ford-Fulkerson algorithm for the maximum flow problem is a special case of Klein's cycle cancelling algorithm, by defining zero costs for all edges in the original graph and by adding an extra edge from the sink to the source with cost -1.

2.1 Choice of cycle Γ

As in the Ford-Fulkerson algorithm, the question is which negative-cost cycle to choose.

1. (Weintraub 1972). One idea is to try choosing the **maximum improvement cycle**, where the difference in cost is as large as possible. One can show that the number of iterations is polynomial for rational costs, but finding such a cycle is NP-hard. For irrational costs, one can show that this algorithm may never terminate (Queyranne 1980) even for the maximum flow problem (the fattest augmenting path algorithm of Edmonds and Karp), although the solution converges to a minimum cost flow.

2. (Goldberg-Tarjan 1986). Alternatively, we can choose the **cycle of minimum mean cost**, defined as follows:

$$\mu(f) = \min_{\text{directed cycles } \Gamma \in G_f} \frac{c(\Gamma)}{|\Gamma|}$$

where $c(\Gamma) = \sum_{(v,w) \in \Gamma} c(v, w)$ and $|\Gamma|$ is the number of edges in the cycle.

Notice that there exists a negative cost cycle in G_f if and only if $\mu(f)$ is negative.

To see that we can indeed find the minimum mean-cost cycle efficiently, suppose we replace the costs c with c' such that $c'(v, w) = c(v, w) + \Delta$ for each edge (v, w). Then $\mu'(f) = \mu(f) + \Delta$, so if $\Delta = -\mu(f)$ then we would have $\mu'(f) = 0$. In particular,

$$\mu(f) = -\inf\{\Delta \; : \; \text{there is no negative cost cycle in } G_f \text{ with respect to costs } c + \Delta\}.$$

For any Δ, we can decide if there is a negative cost cycle by using the Bellman-Ford algorithm. Now, perform binary search to find the smallest Δ for which no such cycle exists. In the next problem set we will show a result by Karp, which finds the cycle of minimum mean cost in $O(nm)$ time by using a variant of Bellman-Ford.

2.2   Bounding the number of iterations

We will give two bounds on the number of iterations for the algorithm. The first depends on the magnitude of the cost and is valid only for integer-valued costs; it is polynomial but not strongly polynomial. The second bound is strongly polynomial and works even for irrational costs.

We first need a measure of 'closeness' to the optimal circulation. The following definition gives such a measure, and will be key in quantifying the progress of the algorithm.

Definition 1 (Relaxed optimality) *A circulation f is said to be ϵ-optimal if there exists a potential $p : V \to \mathbb{R}$ such that $c_p(v, w) \geq -\epsilon$ for all edges $(v, w) \in E_f$.*

Note that an 0-optimal circulation is of minimum cost.

Definition 2 *For a circulation f, let*

$$\epsilon(f) = \min\{\epsilon : f \text{ is } \epsilon\text{-optimal}\}.$$

One important thing about this that we will prove soon is that when we push some flow in a circulation f along some cycle Γ and obtain a new circulation f', we get that $\epsilon(f') \leq \epsilon(f)$. This means that ϵ is monotonically non-increasing in general. First, we need the following strong relationship between $\epsilon(f)$ and $\mu(f)$, and this really justifies the choice of cycle of Goldberg and Tarjan.

Theorem 1 *For all circulations f, $\epsilon(f) = -\mu(f)$.*

Proof: We first show that $\mu(f) \geq -\epsilon(f)$. From the definition of $\epsilon(f)$ there exists a potential $p : V \to \mathbb{R}$ such that $c_p(v, w) \geq -\epsilon(f)$ for all $(v, w) \in E_f$. For any cycle $\Gamma \subseteq E_f$ the cost $c(\Gamma)$ is equal to the reduced cost $c_p(\Gamma)$ since the potentials cancel. Therefore $c(\Gamma) = c_p(\Gamma) \geq -|\Gamma|\epsilon(f)$ and so $\frac{c(\Gamma)}{|\Gamma|} \geq -\epsilon(f)$ for all cycles Γ. Hence $\mu(f) \geq -\epsilon(f)$.

Next, we show that $\mu(f) \leq -\epsilon(f)$. For this, we start with the definition of $\mu(f)$. For every cycle $\Gamma \in E_f$ it holds that $\frac{c(\Gamma)}{|\Gamma|} \geq \mu(f)$. Let $c'(v, w) = c(v, w) - \mu(f)$ for all $(v, w) \in E_f$. Then, $\frac{c'(\Gamma)}{|\Gamma|} = \frac{c(\Gamma)}{|\Gamma|} - \mu(f) \geq 0$ for any cycle Γ. Now define $p(v)$ as the cost of the shortest path from an added source s to v with respect to c' in G_f (see Fig. 1); the reason we add a vertex s is to make sure that every vertex can be reached (by the direct path). Note that the shortest paths are well-defined since there are no negative cost cycles with respect to c'. By the optimality property of shortest
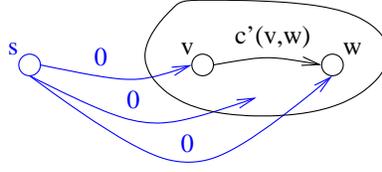
Figure 1: $p(v)$ is the length of the shortest path from s to v.

paths, $p(w) \leq p(v) + c'(v, w) = p(v) + c(v, w) - \mu(f)$. Therefore $c_p(v, w) \geq \mu(f)$ for all $(v, w) \in E_f$ which implies that f is $-\mu(f)$-optimal and thus $\epsilon(f) \leq -\mu(f)$.

By combining $\mu(f) \geq -\epsilon(f)$ and $\epsilon(f) \leq -\mu(f)$ we conclude $\epsilon(f) = -\mu(f)$ as required.   □

The nature of the algorithm is to push flow along negative cost cycles. We would like to know if this actually gets us closer to optimality. This is shown in the following remark.

Remark 1 (Progress) *Let f be a circulation. If we push flow along the minimum mean cost cycle Γ in G_f and obtain circulation f' then $\epsilon(f) \geq \epsilon(f')$.*

Proof: By definition $\frac{c_p(\Gamma)}{|\Gamma|} = \frac{c(\Gamma)}{|\Gamma|} = \mu(f)$. Now, $\epsilon(f) = -\mu(f)$ implies that there exists a potential p such that $c_p(v, w) \geq \mu(f)$ for all $(v, w) \in E_f$. Furthermore for all $(v, w) \in \Gamma$ the reduced cost $c_p(v, w) = \mu(f) = -\epsilon(f)$. If flow is pushed along Γ some arcs may be saturated and disappear from the residual graph. On the other hand, new edges may be created with a reduced cost of $+\epsilon(f)$. More formally, $E_{f'} \subseteq E_f \cup \{(w, v) : (v, w) \in \Gamma\}$. So for all $(v, w) \in E_{f'}$ it holds that $c_p(v, w) \geq -\epsilon(f)$. Thus we have that $\epsilon(f') \leq \epsilon(f)$.   □

2.3 Analysis for Integer-valued Costs

We now prove a polynomial bound on the number of iterations for an integer cost function $c : E \to \mathbb{Z}$. At the start, for any circulation, the following holds for all $(v, w) \in E$:

$$\epsilon(f) \leq C = \max_{(v,w) \in E} |c(v, w)|.$$

Now we can continue with the rest of the analysis.

Lemma 2 *If costs are integer valued and $\epsilon(f) < \frac{1}{n}$ then f is optimal.*

Proof: Consider $-\epsilon(f) = \mu(f) > -\frac{1}{n}$. For any cycle $\Gamma \in G_f$ we have $c(\Gamma) = c_p(\Gamma) > -\frac{1}{n}|\Gamma| \geq -1$. Since the cost is an integer, $c(\Gamma) \geq 0$. By the optimality condition, if there is no negative cycle in the graph, the circulation is optimal.   □

Lemma 3 *Let f be a circulation and let f' be the circulation after m iterations of the algorithm. Then $\epsilon(f') \leq (1 - \frac{1}{n})\epsilon(f)$.*

Proof: Let p be the potential such that $c_p(v, w) \geq -\epsilon(f)$ for all $(v, w) \in E_f$ and let Γ_i and f_i be the cycle that is cancelled and the circulation obtained at the ith iteration, respectively. Let A be the set of edges in E_{f_i} such that $c_p(v, w) < 0$ (we should emphasize that this is for the p corresponding to the circulation f we started from). We now show that as long as $\Gamma_i \subseteq A$, then $|A|$ strictly decreases. This is because cancelling a cycle removes at least one arc with a negative reduced cost from A and any new arc added to E_{f_i} must have a positive reduced cost. Hence after

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$k \leq m$ iterations we will find an edge $(v, w) \in \Gamma_{k+1}$ such that $c_p(v, w) \geq 0$. So by Theorem 1, $-\epsilon(f_k)$ is equal to the mean cost of Γ_{k+1} and thus

$$
\begin{aligned}
\epsilon(f_k) = -\mu(f_k) = -\frac{c(\Gamma_{k+1})}{|\Gamma_{k+1}|} \quad &= \quad -\frac{c_p(\Gamma_{k+1})}{|\Gamma_{k+1}|} \\
&\leq \quad -\frac{0 + (-\epsilon(f))(|\Gamma_{k+1}| - 1)}{|\Gamma_{k+1}|} \\
&\leq \quad \left(1 - \frac{1}{n}\right)\epsilon(f).
\end{aligned}
$$

$\square$

Corollary 4 *If the costs are integer, then the number of iterations is at most $mn \log(nC)$.*

Proof: We have that

$$
\epsilon(f_{end}) \leq \left(1 - \frac{1}{n}\right)^{n \log(nC)} \epsilon(f = 0) < e^{-\log(nC)}|C| = \frac{1}{nC}|C| = \frac{1}{n},
$$

and thus the resulting circulation is optimal. $\square$

The time per iteration will be shown to be $O(nm)$ (see problem set), hence the total running time of the algorithm is $O(m^2 n^2 \log(nC))$.

2.4 Strongly Polynomial Analysis

In this section we will remove the dependence on the costs. We will obtain a strongly polynomial bound for the algorithm for solving the minimum cost circulation problem. In fact we will show that this bound will hold even for irrational capacities. The first strongly polynomial-time analysis is due to Tardos; the one here is due to Goldberg-Tarjan. This result was very significant, since it was the most general subclass of Linear Programming (LP) for which a strongly polynomial-time algorithm was shown to exist. It remains a big open problem whether a strongly polynomial-time algorithm exists for general LP.

Definition 3 *An edge e is ϵ-fixed if for all ϵ-optimal circulations f we have that $f(e)$ maintains the same value.*

Note that (v, w) is ϵ-fixed if and only if (w, v) is ϵ-fixed, by skew-symmetry of edge-costs.

Theorem 5 *Let f be a circulation and p be a potential such that f is $\epsilon(f)$-optimal with respect to p. Then if $|c_p(v, w)| \geq 2n\epsilon$ for some edge $(v, w) \in E$, the edge (v, w) is ϵ-fixed.*

Proof: Suppose (v, w) is not $\epsilon(f)$-fixed. There exists an f' that is $\epsilon(f)$-optimal and $f'(v, w) \neq f(v, w)$; without loss of generality assume $f'(v, w) < f(v, w)$. Let $E_< = \{(x, y) : f'(x, y) < f(x, y)\}$. We can see that $E_< \subseteq E_{f'}$ by definition of $E_{f'}$. Furthermore, from flow conservation, we know that there exists a cycle $\Gamma \in E_{f'}$ containing the edge (v, w). Indeed, by flow decomposition, we know that the circulation $f - f'$ can be decomposed into (positive net) flows along cycles of $E_{f'}$, and thus one of these cycles must contain (v, w)

Now we have the following,

$$
c(\Gamma) = c_p(\Gamma) \leq -2n\epsilon(f) + (n-1)\epsilon(f) < -n\epsilon(f).
$$

Consequently, $\frac{c(\Gamma)}{|\Gamma|} < -\epsilon$ and so $\mu(f') < -\epsilon$. As a result, f' is not $\epsilon(f)$-optimal and thus we have a contradiction. $\square$

Lemma 6 *After $O(mn \log n)$ iterations, another edge becomes fixed.*

Proof: Let f be a circulation and f' be another circulation after application of $mn \log(2n)$ iterations of the Goldberg-Tarjan algorithm. Also suppose that Γ is the first cycle cancelled and p, p' are the potentials for f, f' respectively. From the previous lemma, we have that $\epsilon(f') \leq (1 - \frac{1}{n})^{n \log(2n)} \epsilon(f) < e^{-\log(2n)} = \frac{1}{2n} \epsilon(f)$. Now from the definition of μ we get the following,

$$\frac{c_{p'}(\Gamma)}{|\Gamma|} = \frac{c(\Gamma)}{|\Gamma|} = \mu(f) = -\epsilon(f) < -2n\epsilon(f')$$

This means that there exists an edge $(v, w) \in \Gamma$ such that $c_{p'}(v, w) < -2n\epsilon(f')$ which means that it was not $\epsilon(f)$-fixed. Thus (v, w) becomes $\epsilon(f')$-fixed and the claim is proven. $\square$

Notice that if e is fixed, it will remain fixed as we iterate the algorithm. An immediate consequence of the above lemma then is a bound on the number of iterations in the Goldberg-Tarjan algorithm.

Corollary 7 *The number of iterations of the Goldberg-Tarjan algorithm, even with irrational costs, is $O(m^2 n \log n)$.*