

Lecture 11

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In the last lecture, we saw a cycle canceling algorithm for the minimum cost circulation problem. In this lecture we present a strongly polynomial time algorithm. This note is mainly based on the network flows lecture notes of previous years.

The problem of finding a strongly polynomial algorithm (and even its existence) for the minimum cost circulation problem was open for several years. In 1985, Éva Tardos developed the first such algorithm. In 1987, Goldberg and Tarjan produced an improved version presented below.

1 The Goldberg-Tarjan Algorithm

In the last lecture, we saw the following algorithm.

Goldberg-Tarjan algorithm:

1. Let $f = 0$.
2. While $\mu(f) < 0$ do
 push $\delta = \min_{(v,w) \in \Gamma} u_f(v,w)$ along a minimum mean cost cycle Γ of G_f .

The Goldberg-Tarjan algorithm is a cycle canceling algorithm since G_f has a negative directed cycle iff $\mu(f) < 0$.

2 Analysis of the Goldberg-Tarjan Algorithm

Before analyzing the Goldberg-Tarjan cycle canceling algorithm, we review some definitions.

Definition 1 A circulation f is ϵ -optimal if there exists p such that $c_p(v,w) \geq -\epsilon$ for all $(v,w) \in E_f$.

For $\epsilon = 0$, we know that there exist potential p such that $c_p(v,w) \geq 0$ for $(v,w) \in E_f$, hence a 0-optimal circulation is a minimum cost circulation.

Definition 2 $\epsilon(f) = \text{minimum } \epsilon \text{ such that } f \text{ is } \epsilon\text{-optimal. In other words,}$

$$\epsilon(f) = \min\{\epsilon \mid \exists p : c_p(v,w) \geq -\epsilon \forall (v,w) \in E_f\}.$$

In this section, assume that all $c(u,w)$ are integers. We will see noninteger costs in this lecture note but not in this part. Also we have the following theorems which were proved in the last lecture.

Theorem 1 If f is a circulation with $\epsilon(f) < \frac{1}{n}$ then f is optimal.

Theorem 2 For any circulation f , $\mu(f) = -\epsilon(f)$.

We are now ready to analyze the algorithm. First, we show that, using $\epsilon(f)$ as a measure of near-optimality, the algorithm produces circulations which are closer and closer to optimal.

Theorem 3 Let f be a circulation and let f' be the circulation obtained by canceling the minimum mean cost cycle Γ in E_f . Then $\epsilon(f) \geq \epsilon(f')$.

Proof: By definition, there exists p such that

$$c_p(v, w) \geq -\epsilon(f) \quad (1)$$

for all $(v, w) \in E_f$. Moreover, for all $(v, w) \in \Gamma$, we have $c_p(v, w) = -\epsilon(f)$ since, otherwise, its mean cost would not be $-\epsilon(f)$. We claim that, for the same p , (1) holds for all $(v, w) \in E_{f'}$. Indeed, if $(v, w) \in E_{f'} \cap E_f$, (1) certainly holds. If $(v, w) \in E_{f'} \setminus E_f$ then (w, v) certainly belongs to Γ . Hence, $c_p(v, w) = -c_p(w, v) = \epsilon(f) \geq 0$ and (1) is also satisfied. \square

Next, we show that $\epsilon(f)$ decreases after a certain number of iterations.

Theorem 4 Let f be any circulation and let f' be the circulation obtained by performing m iterations of the Goldberg-Tarjan algorithm. Then $\epsilon(f') \leq (1 - \frac{1}{n})\epsilon(f)$.

Proof: Let p be such that $c_p(v, w) \geq -\epsilon(f)$ for all $(v, w) \in E_f$. Let Γ_i be the cycle canceled at the i th iteration. Let k be the smallest integer such that there exists $(v, w) \in \Gamma_{k+1}$ with $c_p(v, w) \geq 0$. We know that canceling a cycle removes at least one arc with negative reduced cost from the residual graph and creates only arcs with positive reduced cost. Therefore $k \leq m$. Let f' be the flow obtained after k iterations. By Theorem 2, $-\epsilon(f')$ is equal to the mean cost of Γ_{k+1} which is:

$$\begin{aligned} \frac{\sum_{(v,w) \in \Gamma_{k+1}} c_p(v, w)}{l} &\geq \frac{-(l-1)}{l} \epsilon(f) \\ &= -(1 - \frac{1}{l}) \epsilon(f) \geq -(1 - \frac{1}{n}) \epsilon(f), \end{aligned}$$

where $l = |\Gamma_{k+1}|$. Therefore, by Theorem 3, after m iterations, $\epsilon(f)$ decreases by a factor of $(1 - \frac{1}{n})$. \square

Assuming that all $c(u, w)$ are integers, we have the following theorem:

Theorem 5 Let $C = \max_{(v,w) \in E} |c(v, w)|$. Then the Goldberg-Tarjan algorithm finds a minimum cost circulation after canceling $nm \log(nC)$ cycles ($\log = \log_e$).

Proof: The initial circulation $f = 0$ is certainly C -optimal since, for $p = 0$, we have $c_p(v, w) \geq -C$. Therefore, by Theorem 4, the circulation obtained after $nm \log nC$ iterations is ϵ -optimal where:

$$\epsilon \leq \left(1 - \frac{1}{n}\right)^{n \log(nC)} C < e^{-\log(nC)} C = \frac{C}{nC} = \frac{1}{n},$$

where we have used the fact that $(1 - \frac{1}{n})^n < e^{-1}$ for all $n > 0$. The resulting circulation is therefore optimal by Theorem 1. \square

The overall running time of the Goldberg-Tarjan algorithm is therefore $O(n^2 m^2 \log(nC))$ since the minimum mean cost cycle can be obtained in $O(nm)$ time.

3 Cancel and Tighten Algorithm

We can improve the algorithm presented in the previous sections by using a more flexible selection of cycles for canceling and explicitly maintaining potentials to help identify cycles for canceling. The idea is to use the potentials we get from the minimum mean cost cycle to compute the edge costs $c_p(v, w)$ and then push flow along all cycles with only negative cost edges. The algorithm Cancel and Tighten is described below.

Cancel and Tighten:

1. Cancel: As long as there exists a cycle Γ in G_f with $c_p(v, w) < 0, \forall (v, w) \in \Gamma$ push as much flow as possible along Γ .
2. Tighten: Compute a minimum mean cost cycle in G_f and update p .

We now show that the Cancel step results in canceling at most m cycles each iteration and the flow it gives is $(1 - 1/n)\epsilon(f)$ optimal.

Theorem 6 *Let f be a circulation and let f' be the circulation obtained by performing the Cancel step. Then we cancel at most m cycles to get f' and*

$$\epsilon(f') \leq (1 - \frac{1}{n})\epsilon(f).$$

Proof: Let p be such that $c_p(v, w) \geq -\epsilon(f)$ for all $(v, w) \in E_f$. Let Γ be any cycle in f' and let l be the length of Γ . We know that canceling a cycle removes at least one arc with negative reduced cost from the residual graph and creates only arcs with positive reduced cost. Therefore we can cancel at most m cycles. Now $G_{f'}$ has no negative cycles therefore every cycle in $G_{f'}$ contains an edge (v, w) such that $c_p(v, w) \geq 0$. Hence the mean cost of Γ is at least:

$$\begin{aligned} \frac{\sum_{(v,w) \in \Gamma} c_p(v, w)}{l} &\geq \frac{-(l-1)}{l}\epsilon(f) \\ &= -(1 - \frac{1}{l})\epsilon(f) \geq -(1 - \frac{1}{n})\epsilon(f), \end{aligned}$$

□

The above result implies that the Cancel and Tighten procedure finds a minimum cost circulation in at most $n \log(nC)$ iterations (by an analysis which is a replication of Theorem 5). It also takes us $O(n)$ time to find a cycle on the admissible graph. This implies that each Cancel step takes $O(nm)$ steps due to the fact that we cancel at most m cycles and thus a running time of $O(nm)$ for one iteration of the Cancel and Tighten Algorithm. Therefore the overall running time of Cancel and Tighten is $O(n^2 m \log(nC))$ (i.e. an amortized time of $O(n)$ per cycle canceled). We can further improve this by using dynamic trees to get an amortized time of $O(\log n)$ per cycle canceled and this results in an $O(nm \log n \log(nC))$ algorithm.

4 A Strongly Polynomial Bound

In this section, we give another analysis of the algorithm. This analysis has the advantage of showing that the number of iterations is strongly polynomial, i.e. that it is polynomial in n and m and does

not depend on C . The first strongly polynomial algorithm for the minimum cost circulation problem is due to Tardos.

Definition 3 An arc $(v, w) \in E$ is ϵ -fixed if $f(v, w)$ is the same for all ϵ -optimal circulations.

There exists a simple criterion for deciding whether an arc is ϵ -fixed.

Theorem 7 Let $\epsilon > 0$. Let f be a circulation and p be node potentials such that f is ϵ -optimal with respect to p . If $|c_p(v, w)| \geq 2n\epsilon$ then (v, w) is ϵ -fixed.

Proof: The proof is by contradiction. Let f' be an ϵ -optimal circulation for which $f'(v, w) \neq f(v, w)$. Assume that $|c_p(v, w)| \geq 2n\epsilon$. Without loss of generality, we can assume by antisymmetry that $c_p(v, w) \leq -2n\epsilon$. Hence $(v, w) \notin E_f$, i.e. $f(v, w) = u(v, w)$. This implies that $f'(v, w) < f(v, w)$. Let $E_< = \{(x, y) \in E : f'(x, y) < f(x, y)\}$.

Claim 8 There exists a cycle Γ in $(V, E_<)$ that contains (v, w) .

Proof: Since $(v, w) \in E_<$, it is sufficient to prove the existence of a directed path from w to v in $(V, E_<)$. Let $S \subseteq V$ be the nodes reachable from w in $(V, E_<)$. Assume $v \notin S$. By flow conservation, we have

$$\sum_{x \in S, y \notin S} (f(x, y) - f'(x, y)) = \sum_{x \in S} \sum_{y \in V} (f(x, y) - f'(x, y)) = 0.$$

However, $f(v, w) - f'(v, w) > 0$, i.e. $f(w, v) - f'(w, v) < 0$, and by assumption $w \in S$ and $v \notin S$. Therefore, there must exist $x \in S$ and $y \notin S$ such that $f(x, y) - f'(x, y) > 0$, implying that $(x, y) \in E_<$. This contradicts the fact that $y \notin S$. \square

By definition of $E_<$, we have that $E_< \subseteq E_{f'}$. Hence, the mean cost of Γ is at least $\mu(f') = -\epsilon(f') = -\epsilon$. On the other hand, the mean cost of Γ is $(l = |\Gamma|)$:

$$\begin{aligned} \frac{c(\Gamma)}{l} &= \frac{c_p(\Gamma)}{l} = \frac{1}{l} \left(c_p(v, w) + \sum_{(x, y) \in \Gamma \setminus \{(v, w)\}} c_p(x, y) \right) \\ &\leq \frac{1}{l} (-2n\epsilon + (l-1)\epsilon) < \frac{1}{l} (-l\epsilon) = -\epsilon, \end{aligned}$$

a contradiction. \square

Theorem 9 The Goldberg-Tarjan algorithm terminates after $O(m^2 n \log n)$ iterations.

Proof: If an arc becomes fixed during the execution of the algorithm, then it will remain fixed since $\epsilon(f)$ does not increase. We claim that, as long as the algorithm has not terminated, one additional arc becomes fixed after $O(mn \log n)$ iterations. Let f be the current circulation and let Γ be the first cycle canceled. After $mn \log(2n)$ iterations, we obtain a circulation f' with

$$\epsilon(f') \leq \left(1 - \frac{1}{n}\right)^{n \log(2n)} \epsilon(f) < e^{-\log(2n)} \epsilon(f) = \frac{\epsilon(f)}{2n}$$

by Theorem 6. Let p' be potentials for which f' satisfies the $\epsilon(f')$ -optimality constraints. By definition of Γ ,

$$-\epsilon(f) = \frac{c_{p'}(\Gamma)}{|\Gamma|}.$$

Hence,

$$\frac{c_{p'}(\Gamma)}{|\Gamma|} < -2n\epsilon(f').$$

Therefore, there exists $(v, w) \in \Gamma$ such that $|c_{p'}(v, w)| > -2n\epsilon(f')$. By the previous Theorem, (v, w) is $\epsilon(f')$ -fixed. Moreover, (v, w) is not $\epsilon(f)$ -fixed since canceling Γ increased the flow on (v, w) . This proves that, after $mn \log(2n)$ iterations, one additional arc becomes fixed and therefore the algorithm terminates in $m^2n \log(2n)$ iterations. \square

Using the $O(mn)$ algorithm for the minimum mean cost cycle problem, we obtain a $O(m^3n^2 \log n)$ algorithm for the minimum cost circulation problem. Using the Cancel and Tighten improvement we obtain a running time of $O(m^2n^2 \log n)$. And if we implement Cancel and Tighten with the dynamic trees data structure we get a running time of $O(m^2n \log^2 n)$.

The best known strongly polynomial algorithm for the minimum cost circulation problem is due to Orlin and runs in $O(m \log n(m + n \log n)) = O(m^2 \log n + mn \log^2 n)$ time.