PROFESSOR: All right, so lecture 10 was about two main things, I guess. We had the conversion from folding states to folding motions, talked briefly about that. And then the bulk of the class was about Kempe and Kempe's universality theorem and the beginning of linkages. So let's start with an open problem about converting folded states to folding motions. This is a nice question.

So suppose you have a sheet of paper like this one, but it has a hole in the middle. And then you construct some folded state of that piece of paper with a hole in it. And now you'd like to actually get there by continuous folding motion. So the question is why doesn't that work? What we know is that the same proof technique doesn't work. We don't necessarily know that it's impossible. That would be a nice problem to solve, actually.

So what difference does the hole make? So this is the method we saw before. You imagine having some folded state, say, from a flat piece of paper to a crane. You roll up that piece of paper to a tiny triangle that maps to a nice almost flat portion of the crane. Then you basically play that motion backwards, but on the surface of the crane instead of on the flat sheet.

And that always works for simple polygons, polygons without holes. If you have a hole in here, you could imagine just filling the hole. That's what the question suggested. Just fill the hole, do this thing, and then remember that the hole actually wasn't there. Erase it again. That should give you a motion. Erasing the hole is fine.

The trouble is this part. So if I define a folded state of a piece of paper with a hole, it doesn't tell you where-- let's say there's a little hole here-- this mapping won't tell you where that hole goes in 3D. You have no idea. And in fact, it may be impossible to map the hole anywhere in this folded state that's valid.

When you tear a piece of paper, new foldings become possible that were not otherwise possible. What's an example of that? When I do a big tear like this, now I can pull these points of paper apart. And it's impossible to fill this hole in in 3D. It's

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possible to fill the hole in here. It's just suturing. But when I separate things, that means the original sheet could not fold into this state.

So you get new folded states when you have holes, that you cannot just patch the hole and hope to find a place that it folds over here. There are other issues. You could maybe patch it in with some stretchy material or something. I have one example in the notes where you-- suppose you have a tube of paper. So this is not a flat example, but it's an interesting example anyway. So let's say the outside of this tube is purple, and the inside is white.

And one thing you can do with the tube is turn it inside out. So you can make the inside purple and the outside white. So this is possible with a tube of paper. If you imagine this as being a hole and the bottom side is also being a hole-- both of these are open right now-- then you could also imagine filling them in and getting a cube of paper. So in this case you'd get a cube that's entirely purple.

This would be bad, because a cube cannot be turned inside out without self intersection. So this is an example where there is a folding motion without the holes filled in. There is not a folding motion when you fill the holes in. So I don't know what that says exactly about the problem, but it's some intuition why this is tricky business. That's for a polyhedron, of course. If you're just trying to take polygon with holes and fill it in somehow, I mean, maybe there's a way, but certainly the obvious way does not work. What else did have in my notes here?

All right. Any questions about that open problem? Next question is about, it's a neat idea. We talked about linkages that have joints like this one where you must stay connected. And then we briefly also talked about a different kind of joint where this vertex was pinned right along another edge. And we showed you could simulate that by just making a zero area triangle here.

So you can force these edges to come right at this point of that bar. Well, different idea is what if you allow this point to be able to slide along that bar? Is that some new kind of linkage that's more powerful or something? Turns out, no, it's not more powerful. You can simulate that too.

So I thought I'd show that. I had to think about it. It's kind of fun. This would have made a good problem set problem, but I decided to cover it. So remember our good friend the Peaucellier linkage, which looks like this. So these guys are equidistant, and this is a rhombus. All the edge lengths are equal. Then this vertex lies along a straight line. And it has a limit. Let's say it can go this high and this low, something like that. We've seen that in animation before.

So what I'm going to do is imagine this as being my bar, and this as my flexible point that can move along that bar. So here's the existing bar. And if I want to add a point that can' slide along the bar, I'm just going to attach this construction to this bar, and this is where things get a little bit messy.

These are the guys that are normally rigidly on the ground, but instead of them being on the ground, I'm going to attach them here. So I'm going to attach this to that, this to that, this to that, this to that. And because there's two connections, this is at the intersection of two-- I mean, this is a rigid triangle. So these guys can't move anymore. But, well, they move relative to this edge. So however this edge moves in the plane or in space, whatever, I guess in the plane here, this guy will be forced to track along the bar.

So if you don't worry about intersections, which we're not in this lecture, this construction you could attach to any bar and make a point that can slide along the bar. Kind of fun. So you see the power of Peaucellier linkages. You can do all sorts of fun constructions like this. You could make it just occupy a little portion of the bar, whatever you want. Just build the appropriate Peaucellier linkage. OK.

So we're into linkages. Next we're into Kempe's universality theorem, or sort of Kempe's universality theorem that he almost proved. So one question just to review, there is this parallelogram and contra parallelogram. Pretty much all of his construction other than Peaucellier at the end are a mix of these two gadgets. And there's this issue that you could flip one into the other. Here we had, this was the translator gadget. If you have two parallelograms, you can collapse one of them, say, and then flip it out to be a contra parallelogram.

If you don't do any bracing, this can happen. And this is bad. You can actually see a few ways in which this is bad. One is here, we have the parallelogram-- or, the point of this translator gadget was to preserve this angle alpha, the green angle here. It's supposed to be the same as the green angle here. But if you do this flip, it won't be anymore. It's the angle between this and horizontal. Right now this edge is almost horizontal. So the new angle's almost 0. Here it's not.

Here's an example with contra parallelograms. This is our angle trisector. If you line up this big angle with something, you get the thirds of it, or vice versa. We actually wanted to use it to triple angles. This is Kempe's original drawing. And if, say, this outermost contra parallelogram flipped open and became a parallelogram like this blue, then you'd be in big trouble. These two angles no longer equal this third angle. So you'd no longer be tripling or trisecting. So that's why it's bad.

Next question is, how did you fix it again? I mean, the parallelogram was easy. I don't think I need to review that using the construction we already talked about. Let me tell you a little bit about the contra parallelogram bracing, although it's very messy to prove that this works, so I don't want to spend too much time on it. The idea again was to take the midpoints of the four edges of the contra parallelogram. And first you prove those always remain colinear in the contra parallelogram state. They are not colinear in the parallelogram state, and that's kind of what's good about it.

Then you find a magic point out here off the board called X, and X is going to be on this perpendicular bisector of PR. It's also the perpendicular bisector of SQ. Turns out this distance always equals this distance by the symmetry of contra parallelogram. That's actually really easy to see, because you have opposite edge links being equal. You get that symmetry.

So it turns out that has to be a fairly specific point for this to work. All right. So what do we do next? And then we add these four bars. And the harder part of the claim is at that this thing still moves within an appropriately chosen X. I kind of don't want to get into that too much. The easier part to see is that you can no longer-- if X is

sufficiently far down there, this is no longer possible. So let's prove that first.

So let's see. If you look at these bars, the bar PX and RX have the same length. That means however you fold this thing, X must be on the perpendicular bisector of PR. Here, that's fine. Over here, the perpendicular bisector would be, I guess, some thing like this, I guess. All right? So the perpendicular bisector of PR some ray like that. OK.

And simultaneously for the same reason, X must be on the perpendicular bisector of S and Q. S and Q are these opposite midpoints, and so you've got to be in some kind of perpendicular bisector here. If you have to be on both of those lines, that means in fact you must be at the center of this parallelogram. Or, I don't really need this as a center. It's some point inside the parallelogram.

That's really bad for X. If these links are really long, say, longer than the perimeter of this linkage, then X has to be outside, because, yeah. If it has to be far from S, R, Q, and P, you can't be inside the polygon. OK? So provided these lengths are sufficiently long, say longer than the perimeter, there's no way X is inside, and yet in the parallelogram state it has to be inside. And so the parallelogram stays impossible.

So that's the easy part of the proof. The tricky part is to get this thing to still fold when this is in the contra parallelogram state, that X is still OK here. And I'll just mention to convince you that it's tricky-- you set the length of the XS bar must be in square, this thing. XS bar squared must be the XP bar squared plus 1/4 AB squared minus AD squared. And I won't go into the proof. There's some details in the notes here.

But what this says is that we're talking about XS versus XP. The other two are symmetric , so they have to be equal. So XS has to be a bit bigger than XP, and this says how much bigger. The formula says how much bigger. They can both still be very large, so we can still get the part that we need, but we need that they're actually related in this way for the whole thing to hold together. And I will leave it at that.

Sorry, it's a little unsatisfying, but the details are just not that exciting. If you're interested in them, you can read Tim Abbot's master's thesis, which is on my web page. Cool. I wanted to briefly remind you about some of the project ideas for Kempe before we go to generalizations of Kempe. So one of them is to implement Kempe. It's never been implemented, as far as I know, in general form. And it would be interesting to actually see it happen in action, some version of it. There's a lot of different versions, but ideally with bracing, I guess.

Another fun sort of more design project would be to design an alphabet and be able to make every letter of the alphabet with some linkage. That doesn't have to follow Kempe, but it would be in the spirit of signing your name. And I have here one example that's on the web. I'll show you the web page of making the letter C. Here's what it looks like in action.

So it's just a four bar linkage, pretty simple. I mean, it's three bars plus this closing bar. And then you look at the midpoint of this edge, and it happens to trace out this kind of letter C. So if you had a pen there, that's what it would make. You could imagine just a whole bunch of these in sequence and be another kind of mathematical font which would be fun to have, so. A lot more, 25 open problems left to go. I think I could do a circle. I can do an O, so. 24, and these [? fall ?] fast.

Another direction would be to build some kind of sculpture inspired by Kempe. This is one by Arthur Ganson since called *Faster!* And if you've been to the MIT Museum, you may have seen it. Sometimes it's out, although I've never seen it running. But there's a video of it online if you want to check it out. So this is a device. It's a kind of push cart. It's a sculpture push cart. You have to run with it, and as you run, the wheels power these gears, and the pen there with the hand signs faster, as in you should push it faster.

It's pretty crazy. Now, this could be done with Kempe, but in this case it's not. It's done with these weirdly shaped gears. And those weirdly shaped gears control the different axes of the pen. X, Y, Z, N, in and out. You can see it actually lifts up to do the exclamation point. So that's one sculpture inspired by Kempe. In general, there

are lots of ideas for making sculptures out of linkages.

Arthur Ganson is particularly cool in making linkage-like sculptures that move kinetically. If you haven't been to the MIT Museum to see his stuff, you really should. It's super cool. All right. So that's that. Next we go on to generalizations of Kempe. So there are few questions about this. One was higher dimensions, how exactly does that work? Particularly D equals 3. If you want to follow a surface, does that mean the linkage now has two degrees of freedom? The answer is yes.

You get to choose. If you want to trace out a surface, you'll have to have two degrees of freedom. So it's not just turning a circular crank. It's more like a spherical crank, which is indeed what my hand can do relative to my elbow is move along a sphere, well, maybe a half sphere or something. So that's what's possible there. If you want to trace out a 3D curve, then you only have one degree of freedom, of course.

I thought I'd briefly tell you a little bit about how 3D works, or show you one of the constructions, which is the Peaucellier linkage. So here is the 2D Peaucellier linkage. In 3D, this won't work. Well, it won't work in the sense that this guy is rather unconstrained. But if we add, let's see, it's a little hard to see. Imagine a plane here, a vertical plane through these points. So if this is in the board plan, I want to choose a point that's roughly here out of the plane. I'm going to draw that here. And then connect it up the same as before.

So it's connected to here. It's connected to here. And it's connected to there. So it's just a third point just like these two. These two are symmetric, so this one's also symmetric. And all these lengths are equal if you put it at right C coordinate. Then this is like a 3D Peaucellier. I don't think Peaucellier invented it. Probably we did, but the result is that this point will lie on a plane out here. So that's cool. It's like the higher dimensional version of Peaucellier.

Now if you actually want this point to move along a line, what would you do?

AUDIENCE: Intersect two planes.

PROFESSOR: Intersect two planes, exactly. Take two planes, intersect them. It is a line. There's the line of intersection. Generically, you get a line. Unless they're the same planes, you always get a line. So if you take two Peaucellier linkages, you overlap them at this one point. Then this point, on the one hand, these two points are pinned. We'll have to line one plane. On the other hand, I'll have to line another plane from the other Peaucellier linkage, so you can force it to stay on an actual line.

And both of these gadgets are useful. Sometimes you want things to stay in planes. Sometimes you want things to stay on lines. And once you have this construction, in fact, you could build the old Kempe construction, and just put in a ton of 3D Peaucellier linkages to force everything to stay in the XY plane. And then anything you could do in two dimensions you can now do in three dimensions. So that's observation one.

You can do Kempe in the Z equals 0 plane, the XY plane. And that's the basic idea for 3D. You do all the stuff we're used to doing in there, all the angle doubling, adding new addition, all these things, and then you just have to translate. You have some points in 3D. You need to measure their coordinates or the angles they form with other edges in 3D. You just map all those things into the XY plane, do your computation on the XY plane, and then map them back. I won't talk about that mapping, but it's not too hard.

And once you can map anything you want into the XY plane, you could do your computation, map it back, and force your points in three dimensions to have whatever properties you need. So you can write down any polynomial now in X, Y, and Z and set that equal to 0 just like before. Do all the trig expansions you did before, and you can force a point to trace exactly that curve in 3D.

So that's just a sketch of how 3D works. Skipping some details because there are messy. The idea is very simple. 3D Peaucellier. All right. Next question is related to-- properties is mentioned in the lecture notes. So there's a couple versions of this question. They're asking the same things at different levels of detail. So what about curves not represented by a polynomial?

I read this like, well, that's not possible. Everything you make out of linkages has to be represented by a polynomial. That's true, but what about piecewise polynomials? That is possible. So you can do piecewise cubic splines. If you've ever drawn a curve in a vector drawing program, you've used splines.

So those are splines, and they're made up of little polynomial pieces, like maybe of a parabola here, and then you design it to transition say, C2 into another parabola, or then into some hyperbola, whatever. So these kinds of general curves, you can do great things with splines. Pretty much every curve you've seen on a computer is a spline.

And this is much better than the Weierstrass approximation theorem which we talked about before, which says you design one polynomial that approximates an entire curve, like your signature. But when it does that, it'll be like this. It's a very ugly-- if you use various pieces of polynomials, say all cubic polynomials, you can get some really nice looking curves. So you can really reproduce your signature.

And there is a theorem mentioned in the notes that says you can trace any semi algebraic set. So I want to define semi algebraic, because this is actually closely related to splines. It's a little more general. So what's a semi algebraic set? So here's an example of a semi algebraic set. You have some polynomial, let's say on XYZ, and you want to say this is greater than or equal to 0.

So it's a little different. Before, we could set polynomials equal to 0. Now I can set them greater than or equal to 0. So that's the semi part of-- if we just have this, this is an algebraic set, essentially. Semi algebraic, you can have half spaces in some sense. And then you can also take unions and intersections to form an algebraic set. Also compliments, but it doesn't matter. So what does this mean? It means I can take all the stuff on one side of a polynomial, and then-- well, on one polynomial. Sorry, that's what I should do.

So here's, let's say, my parabola. I can take this region, and then I could say, OK, well, let's take-- this gets messy to do-- I could take also all the stuff outside this

polynomial. So that's some bigger region here. I could take the union of those. I could clip off parts. Basically I can construct a spline in particular, but in general I can do lots of different things by unions and intersections of these polynomial half spaces.

So this lets you construct splines. It lets you piece together components, because, for example, I could take this curve. And then on one side intersect it with the other side Greater equal to 0 and lesser equal to zero, then I get exactly just this curve. And then I can, for example, cut to the left of this line. And then I'll have this curve, but only if it stops here. I could do the same thing with this piece, end up with this piece, and then take the union of those two pieces. So I can construct a spline.

I can have regions, of course. Infinite area, finite area, whatever. So semi algebraic sets are very general. It's fairly easy to see that this is the most you could hope for, because in general, you look at a linkage, it's defined by polynomial equations. You say, well the square distance from P to Q equals L for various things. So you only have polynomial equations to work with. Because you have flexibility, you get inequations. Because we can do things like this. I mean, this distance is now less than or equal to the sum of these two lines.

So semi algebraic sets are the best you could hope for, a bunch of polynomial inequalities. And in fact, every such semi algebraic set is possible. So how do you prove that? It's actually really easy. We've essentially already done P of XYZ is greater or equal to 0, because we saw in Kempe how to set something equal to 0.

To do that, we used the Peaucellier linkage, which I will draw for the Nth time, because I need to modify it. So here's a Peaucellier linkage. It forces this point to lie on a straight line. If I add a joint here, and basically let this length get smaller if it wants to-- it can do things like this now-- I should draw it more like that. Maybe I should draw it scale, so that's maybe something like this. This point can now move anywhere on the segment.

Then this guy will end up being able to make, well not exactly this half plane, but a region of it. A big enough region if you set it up right. And as you may recall, the x-

coordinate here was the sum of all my trig terms. And I wanted that equal to 0 for this to happen, but if I wanted to make greater than or equal to 0, then I just needed to be to the right of that vertical line. And so that lets me take any polynomial and set it greater or equal to 0. So that's basically Kempe, except I used this modified Peaucellier that let's things go to the right a little bit.

OK. Intersections are also easy. If I want to take the intersection of two sets, I just overlay those linkages and let them share the same point. I call this P that we're constraining. So that will apply multiple constraints to that same point, and so that is the intersection of those two semi algebraic sets. The one tricky part is unions, and this is what the question is asking about here.

Intersection gadgets are clear, but what's the union gadget? Union gadget turns out to also be possible using Kempe. Kind of surprising. Let me show you how. So suppose you have linkage one constraining some point I'll call it P1 here. And you have another linkage two, constraining at some point P2. And what you'd like to do is build a new linkage that has some point P that either follows P1 or follows P2, so it takes the union of those two sets. So whatever L1 constructs for P1, whatever L2 constructs for P2, you want to be able to trace P1 or trace P2.

And what we're going to do is build another box here, another linkage, which is going to be-- I won't write L. Got some more room here. It's going to be a Kempe construction for this polynomial. So this polynomial, it's a little different from what we've seen. And over here it's going to be point P. P here is X comma Y. P2 is X2Y2 and so on. If you want, as X1Y1.

So this is an equation involving three points. In the past, we've only had polynomial equations involving one point. This is another generalization of Kempe which I may or may not have mentioned, but it's really easy. We had, what was it, a rhombus to represent a single point. You just have three of them, and now you've got three points.

You could do the same trig identities, and so on, to expand out. You get, you might not call it a polynomial, you might call it a multinomial. Well, I guess it was already a

multinomial in X and Y. Now it's a polynomial on XY, X1, Y1, X2. Y2. And that's exactly what we have here, various powers of those six variables.

You expand them out just in the way you did before with trig, and you just need to be able to add up angles now, not just alpha and beta, but now there's six possible angles representing each of the x- y-coordinates. So I claim I can build this thing of Kempe. OK. And if I build this thing, basically I force X and Y to be either X1, X2 or Y1, Y2, and that lets you take the union of link [? which ?] L1 and L2. The new point P can either live at P1 or can live at P2. Questions about that? Yeah.

AUDIENCE: Should some of those be pluses?

PROFESSOR: Should some of those be pluses?

AUDIENCE: [INAUDIBLE] looks like X equals X1 is enough to make that whole thing 0.

PROFESSOR: Yeah. I was curious about this. Do you see a way to fix this? The worry here is X could be X1, and Y could be Y2 [INAUDIBLE]. So you have--

AUDIENCE: X minus X1 squared plus Y minus Y1 squared all [INAUDIBLE] X minus X2 squared plus Y minus Y2 squared.

PROFESSOR: Yes. Plus-- this is extremely ugly. OK, let me maybe rewrite it. Thanks. Yeah, as I was writing this I was like--

AUDIENCE: Squared in the wrong place.

PROFESSOR: Squared plus Y minus Y1 squared. And then this thing times yeah, thanks. Same thing with 2's. Equals 0. So the product being equal to 0 means one of the two terms better equal 0. And in this case, if this equals 0, because of the squares it forces it to be non-negative, which means the only time the sum is equal to 0 is when both of the terms are equal to 0, which means X equals X1 and Y equals Y1. So this plays the role of and here, and the product plays the role of or. So either XY equals X1Y1, or XY equals X2Y2. Thank you. Good fix.

Just make a quick note of that. Do I have a pen? I won't. OK, so that is how you do

intersection of two linkages, is just Kempe again. It's kind of cool. Any questions about that? Once you have unions, we've already done intersections in half spaces, then you can make any semi algebraic set. So what I think would be cool in particular is to implement Kempe force spline, say, quadratic splines.

Because Kempe for quadratic polynomials is going to be pretty reasonable. You have to implement this union gadget to piece together the pieces, which is kind of messy, but in principle, you can piece together a bunch of polynomials and see what a spline looks like. You had a question.

AUDIENCE: So if the curves you're taking in enough don't intersect to get from point to the other--

PROFESSOR: OK. Yeah. So you're asking about sort of continuous crank ability of Kempe constructions. And indeed, if you had two sets that were disjoint, there's no way to continuously go from one spot to the other. What this is saying is that the overall trace of this point-- if you look at all possible configurations, the linkage, and then just see where P goes, it will trace out both of those connected components. On the other hand, if these do overlap if there is a common point between these two linkages, then this will allow you to transition from one to the other.

So right, if you're building a spline, presumably you know that they're connected, and you want to be careful of the way you union them together to make them possible by continuous motion. I think that's always possible.

But you do have to be careful. Other questions? All right. I want to get the hypar gluing, but there's one more topic some people asked about, which are these origami axioms. I mentioned them very briefly at the end of lecture 10, and they sounded amazingly powerful. They let you solve all these things. The setting here is something called ruler and compass constructions.

How many people here have heard of ruler or straight edge and compass? Almost everyone. A compasses is this gadget like this. You can draw circles with it. And there's a standard mathematical formulation of a straight edge and compass which

is, if I have two points, I can draw a straight line through them. If I have two points, I can draw a circle through them like that. And if I have lines and circles, I can take their intersections.

And if that's all you're allowed to do, then you can prove that if you look at the coordinates of all the points you make-- and let's say you start with one point which I'll call 0 comma 0, another 0.1 comma 0. So I have the number 0 in one. Then the numbers you can make, the coordinates you can make, are everything you could make from 0 to 1 by plus, minus, times, divide, and square root. You could do all those operations, and that's all you could do.

And so what that means basically, is you could solve quadratic polynomials, but nothing more. And that's an old result from 1800s and implies things like you cannot trisect an angle. You can bisect an angle because that only involves quadratic stuff. You can't trisect an angle, for example, 60 degrees, because that involves solving subcubic, which is not possible by straight edge and compass. You cannot compute the cube root of 2. It's a cube doubling problem, so all these great things.

Then came along Huzita in 1989, and at the same time, Jacques Justin in 1989, who you remember did Kawasaki's theorem and Maekawa's theorem. He also did this, all independently. So Huzita suggested these axioms for folding. If I have two points, I can fold a crease along those two points, that line. If I have two points, I can fold a point onto a point. That constructs a perpendicular bisector. If I have two lines, I can fold one line onto the other. That is the angular bisector.

If I have a point and the line, I can fold the line onto itself , which forces the crease to be perpendicular and pass through this point. You see these in lots of origami diagrams. They let you find interesting lines. If I have two points on a line, I can fold this point onto this line while also folding through this point. That's a tangent of a parabola if you look at it correctly.

And if I have two points and two lines, I can fold this point onto this line while simultaneously folding this point onto this line. There actually is four or eight different ways to do it in general, but you can find them all by just manipulating the

paper. That's the claim. There's one other that Huzita missed, Justin saw, called, these days, Hatori's axiom, where you fold a point onto a line. And not shown here, I believe, is another point that you must fold through.

No, that looks like this axiom. So I've forgotten what the differences is for Hatori. Oh, not drawn here is, there's an edge in the bottom. And you also want to fold the line onto itself, which means you have to be perpendicular to this black line down here. So that's sort of all you could imagine if you have these sort of points onto lines, lines onto points, lines onto lines.

Axioms, if you enumerate them all, this is all of them for a single fold. And with these axioms you could prove you can solve any cubic polynomials. So basically you can also take cube roots. And so in particular, you can do things like trisect an angle. It's what's shown up here. Fairly small sequence. This was discovered in the 1970s. You can trisect any angle, divide it into thirds just by these kinds of folds. And the tricky operation here is folding two points onto two lines simultaneously.

That's the third degree operation. Everything else you can do with ruler and compass. And you can also do things like double a cube. This is computing a cube root of 2 ratio. First you bisect you thing into thirds. This is easy to do, because we can divide by three. Then you fold these two points onto these two lines. And it turns out the ratio here between this y-coordinate and this y-coordinate is cube root of 3. A over B is cube root of 3.

This is by Peter Messer in 1985. So that's kind of cool. But that's all you can do with single folds. The most you hope for is solving cubic equations. You can't quintisect an angle. You can't divide an angle in fifths. You can't compute the fifth root of 2, I assume. You can't do lots of things.

But it's at least more powerful than straight edge and compass, which is cool. So then this was just for single fold operations. You could look at two fold operations, three fold operations where you make two folds and simultaneously align lots of things. With two folds you can-- oh, sorry, before I get that. There is a software called ReferenceFinder by Robert Lang, if you're curious about how to construct

various things.

So here is plugged in, I want to compute a third. And it just enumerates all possible things you could do with five or six folds. And if it finds an exact solution, it will put it at the top. It also finds approximate solutions which are practically useful. So this is a sequence of operations that from a square, you could find this point at 0 comma third, which is nice.

When you have two folds, you can quintisect an angle. These are Robert Lang's diagrams for quintisecting a given angle. You can see at the very end here we have an angle and it's divided evenly into fifths. It's a bit complicated, and at some point it involves a two fold operation. It says here. Here's where it all happens. Your fold here, and simultaneously you fold here. And you have to align all these points and lines and things.

And that's cool. I think with three folds, they can solve any quintic equation. That's Alperin and Lang. And then the culmination, which I mentioned briefly at the end of lecture 10, is if you allow end folds simultaneously, then you can solve a degree end polynomial [INAUDIBLE] order N. First you set up your piece of paper into all these independently manipulatable limbs. You mark off these coordinates, which are the lengths of the bars in your Kempe construction. And then you say, well you've got to fold so that this all these points align.

That will construct a linkage state, and if you set Kempe up right, there will only be one state, which is the solution to your polynomial. So that's one way to do it. There are actually other ways to do it. Alperin and Lang have another solution. It's a little more simple, but this is fun because it uses Kempe. And that is a brief story of origami axioms. Any questions about that?

There's a small chapter in the book. It's called Geometric Construction. I didn't prove anything here, because it's a little bit tedious to prove. You can solve any cubic, you can solve all these things. That's the most you could solve. But all these things have been completely characterized. I guess there isn't a complete characterization of, say, two fold axioms, exactly what you can make. That's still

open.

But the point is, as you add more folds, you get more power. Eventually you get all polynomials, which is the most you could hope for, for any kind of geometric construction. If there are no more questions, we resume our task of building something out of hyperbolic paraboloids. Remember this is the hat construction.

If you have a square in your polyhedron, you take four hyperbolic paraboloids and join them together like this picture. And then this is going to represent one edge. Those two edges of the hypar are going to represent one edge of the square. And I was suggesting we make this shape, which we have enough hypars for. We just need to do some taping.

This is the truncated tetrahedron. It's got four triangles, four hexagons. So we've already made-- last time we made the four triangles. These are the three hats, and they can be joined together to form a tetrahedron by themselves. We've got enough hypars to make the four hexagons, and then we just need to tape them together, and we will get the truncated tetrahedron. Who would like to help? Come on up.

AUDIENCE: [INAUDIBLE].

AUDIENCE: [INAUDIBLE].

AUDIENCE: Yep.

PROFESSOR: It will be something like this.

AUDIENCE: OK.

PROFESSOR: I think we need a whole other six, but.

AUDIENCE: [INAUDIBLE] Is there another roll of tape hidden somewhere?

                        [INAUDIBLE].

STUDENT: And the other main thing is [INAUDIBLE].

STUDENT: [INAUDIBLE] these two on a triangle here. It's like 36. Sort of.