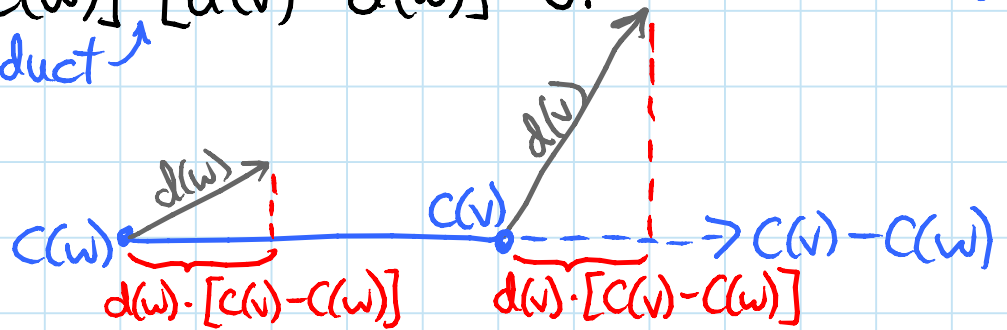<u>Infinitesimal rigidity</u>: rigidity to the <u>first order</u>

<u>Infinitesimal motion</u> of a linkage configuration C
   $=$ valid first derivative of a motion w.r.t. time, at time $\emptyset$
   $=$ <u>velocity vector</u> $d(v)$ for each vertex v
   preserving edge lengths to the first order:
$$[C(v) - C(w)] \cdot [d(v) - d(w)] = 0.$$

$a \cdot b \longrightarrow$ dot product
$= a_x \cdot b_x + a_y \cdot b_y$

$d(w)$
$d(v)$
$C(w)$
$C(v)$
$C(v) - C(w)$
$d(w) \cdot [C(v) - C(w)]$
$d(v) \cdot [C(v) - C(w)]$

<u>Rigidity matrix</u>: "everything is linear, to the first order"
   edge-length constraints form a linear system:

row per edge $\left\{ \begin{pmatrix} 0 \cdots 0 \mid C_x(v) - C_x(w) \mid C_y(v) - C_y(w) \mid 0 \cdots 0 \mid C_x(w) - C_x(v) \mid C_y(w) - C_y(v) \mid 0 \cdots 0 \end{pmatrix} \cdot \begin{pmatrix} d_x(v_1) \\ d_y(v_1) \\ \vdots \\ d_x(v_n) \\ d_y(v_n) \end{pmatrix} = \emptyset \right.$

dn columns (d dim., n vertices)

RIGIDITY MATRIX R

<u>Infinitesimal motions</u> $= d$ for which $R \cdot d = \emptyset$
                     $=$ <u>kernel</u> $R = $ <u>nullspace</u> R
   $\hookrightarrow$ linear subspace of some dimension: <u>nullity</u> R
<u>Infinitesimally rigid</u> if nullity $R = \dfrac{d(d+1)}{2}$ $\leftarrow$ rigid motions

Rank−Nullity Theorem: rank R + nullity R = # cols. = $d \cdot n$

$\Rightarrow$ inf. rigid $\Leftrightarrow$ rank $R = d \cdot n - \frac{d(d+1)}{2}$ "full rank"

$\Rightarrow$ can test inf. rigidity in polynomial time
using e.g. Gaussian elimination

Generic point set (more explicit definition than L9)

= all minors of rigidity matrix of complete graph
   induced square submatrix on subset of rows & cols.
   with nonzero determinant for some point set
   (i.e. not identically zero, algebraically)
   are nonzero for this point set

Generic results:

− almost every configuration is generic
− at generic configurations,
   rigidity = infinitesimal rigidity = generic rigidity [L9]
$\Rightarrow$ randomized polynomial−time generic rigidity test:
   test infinitesimal rigidity of random realization
− if any realization is infinitesimally rigid
   then graph is generically rigid
   (else generically flexible with probability 1)

Taking derivatives: flexible $\Rightarrow$ infinitesimally flexible

   i.e. infinitesimally rigid $\Rightarrow$ rigid
− but not vice versa:
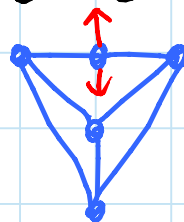
infinitesimal motion   rigid

Tensegrity = tens(ional int)egrity [Snelson 1968;
                                    R. Buckminster Fuller]
PROJECT: build tensegrity sculpture
= linkage but where each edge is either:
  — bar (as before): fixed length
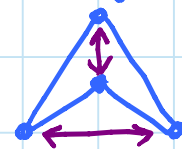 — cable: length can only decrease (string/
  — strut: length can only increase   elastic/
                                       spider web)
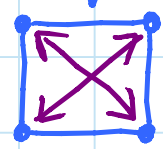
— configuration space becomes semi-algebraic
— motion, rigidity as before   polynomial inequalities
— but not generic rigidity:

flexible   rigid

— infinitesimal motion (& rigidity):
    $[C(v) - C(w)] \cdot [d(v) - d(w)] = \emptyset$ for bars vw
                                      $\leq \emptyset$ for cables vw
                                      $\geq \emptyset$ for struts vw

$\Rightarrow$ inf. motion space is a polyhedral cone
$\Rightarrow$ inf. rigidity testable in polynomial time
       via linear programming

Equilibrium stress = real number for each edge $s: E \to \mathbb{R}$

such that $s(e) \geq 0$ for cables e (push back in resistence)

$s(e) \leq 0$ for struts e

EQUILIBRIUM $\to$ $\displaystyle\sum_{w: \text{edge } vw} s(vw) \cdot [C(v) - C(w)] = 0$

for all vertices v.

— view $s(vw)$ as a <u>scale factor</u> on force along edge vw felt equally by v & w.

     — $s(vw) > 0$ $\Rightarrow$ push on v & w (resist compression)

     — $s(vw) < 0$ $\Rightarrow$ pull on v & w (resist expansion)

     — $s(vw) = 0$ $\Rightarrow$ no force

— <u>trivial equilibrium stress</u>: $s(e) = 0$ for all e

Example:

equilibrium stress | force polygon | no nontrivial equi. stress
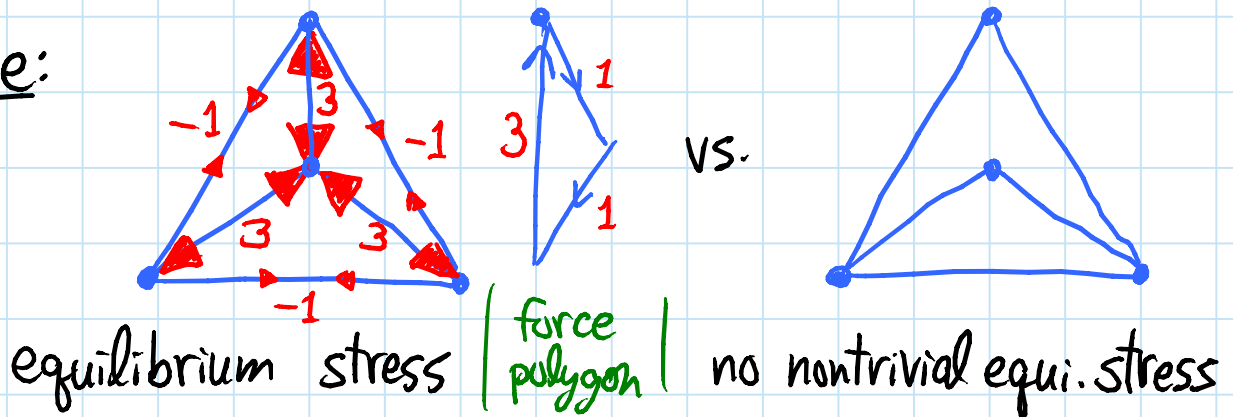
Duality in tensegrities: [Roth & Whiteley 1981]

— some equilibrium stress is nonzero on strut/cable e

$\Leftrightarrow$ every infinitesimal motion holds e's length fixed

— tensegrity is infinitesimally rigid

$\Leftrightarrow$ every strut/cable is nonzero in some equilibrium stress

& corresponding <u>linkage</u> is rigid

     $\hookrightarrow$ replace cables & struts with bars

} not necessary for "spiderweb" (next page)

— proofs based on linear-programming duality

<u>Spiderwebs</u>: all-positive equilibrium stress
    (except on boundary)
⇒ infinitesimally rigid [Connelly 1982]
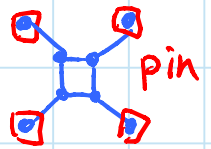
<u>Origami tessellations:</u>
—much history [Momotani 1984; Fujimoto 1982;
    Huffman 1960s, 1978; Resch 1968; Barreto 1997;
    Palmer 1997; Bateman 1990s; Verrill 1990s;
    Lang 2000s; Gjerde 2009]
—<u>shrink-rotate algorithm:</u>

⓪ tessellate ① shrink ② rotate ③ fill ④ fold
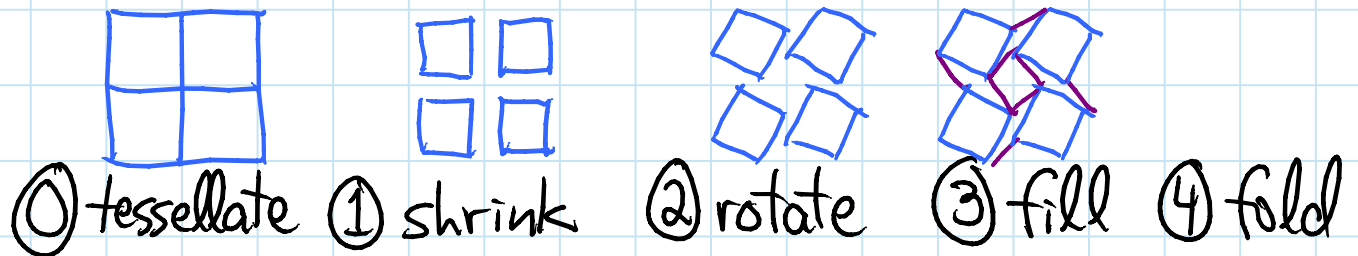
—works for some shrink/rotate amounts
  ⟺ tessellation is a spiderweb
    [Lang & Bateman 2010]

Polyhedral lifting of a noncrossing configuration

= z coordinate for each vertex $z: V \to \mathbb{R}$
such that each face remains planar
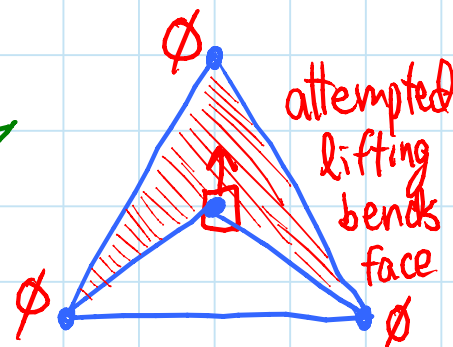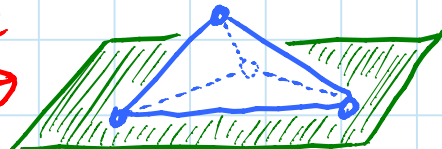- assume outside face at $z=0$ by rigid motion
- trivial lifting: $z(v) = \emptyset$ for all v

Example:
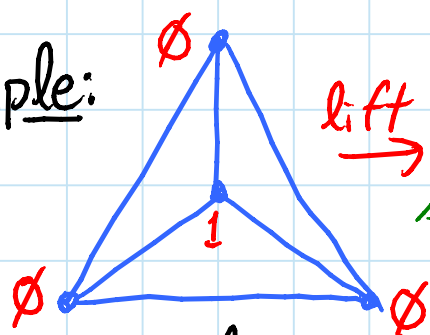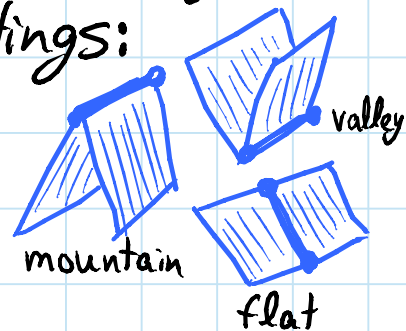

polyhedral lifting   no nontrivial poly. lifting

Maxwell-Cremona Theorem: [Maxwell 1864; Cremona 1872]

one-to-one correspondence in noncrossing tensegrity between
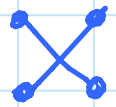equilibrium stresses & polyhedral liftings:
- negative stress $\Longleftrightarrow$ valley edge
- positive stress $\Longleftrightarrow$ mountain edge
- zero stress $\Longleftrightarrow$ flat edge

PROJECT : implement program to illustrate
stress $\Longleftrightarrow$ lifting correspondence
and/or stress $\Longleftrightarrow$ inf. motion correspondence

PROJECT : virtual tensegrity building toy
- illustrate infinitesimal flexibility if any

Noncrossing linkages:
- _configuration_ cannot have crossing edges
- _config. space_ smaller; still semi-algebraic

Locked linkage if config. space is disconnected
   i.e. no motion between some two configurations
- summary:

| [L1]	chains	trees
2D	never locked	can lock
3D	can lock	can lock
4D+	never locked	never locked

Carpenter's Rule Theorem: [Connelly, Demaine, Rote 2000/2003]
any linkage configuration of maximum degree 2
has a motion that
- straightens/convexifies all _outermost_
 open/closed chains
- is _expansive_: distance between any two vertices
   only increases
⇒ is noncrossing, by △ inequality

outermost
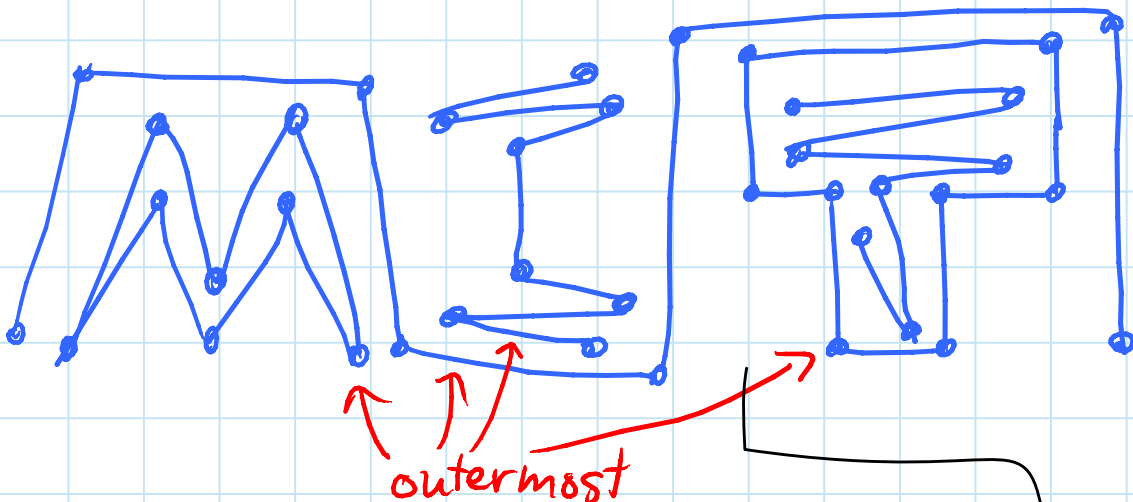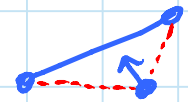
Proof sketch of Carpenter's Rule Theorem:

- build tensegrity from linkage (edges → bars)
    + all possible struts (except where bar exists)
- linkage has expansive infinitesimal motion
  ⟺ tensegrity is infinitesimally flexible
  ⟸ every equilibrium stress is zero (on struts)
  ⟺ every polyhedral lifting is flat (on struts)
    - detail: need to show stresses are equivalent
        in tensegrity vs. planarized tensegrity
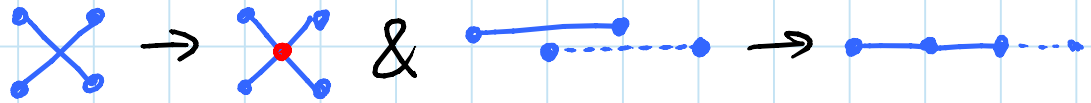
- here is where nested components get discarded
- <u>slice</u> hypothetical polyhedral lifting near max. z:
    - <u>peak case</u>:

slice is polygon
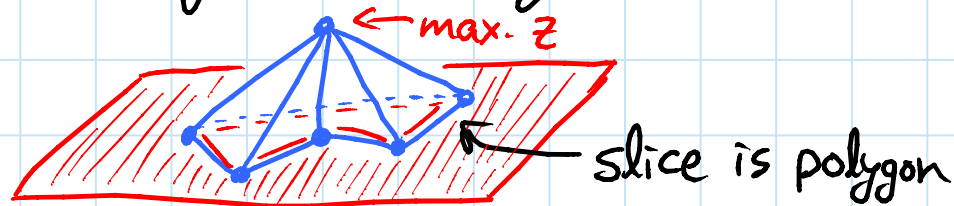
- convex vertices ⟺ mountain edges
- reflex vertices ⟺ valley edges
- every polygon has ≥3 convex vertices
⟹ ≥3 incident mountains (positive stress)
⟹ ≥3 incident bars (no cables)
- but max. degree 2
    - <u>general case</u>:

flat: need convex vertex
to reach around reflex angle

⟹ flat except inside convex polygons
- integrate ordinary differential equation → expansive motion

6.849 Geometric Folding Algorithms: Linkages, Origami, Polyhedra
Fall 2012