6.777J/2.751J Design and Fabrication of Microelectromechanical Devices
Spring Term 2007

Massachusetts Institute of Technology

PROBLEM SET 7 (18 pts)

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Posted 4/20/07

Problem 15.7 (4pts): Design of a PID controller

a). From Eq. (15.5), the transfer function after normalizing the frequency to the undamped resonant frequency is,

$$\frac{X(s)}{F(s)} = H(s) = \frac{1/k}{\hat{s}^2 + \frac{1}{Q}\hat{s} + 1},$$

where we have,

$$\hat{s} = \frac{s}{\omega_o}, \; where\; \omega_o = 2\pi \cdot 50kHz = \pi \times 10^5 rad/s$$

$$k = \omega_o^2 m = \pi^2 \times 10^{10} (rad/s) \times 10^{-9} (kg) = 10\pi^2$$

$$Q = 10$$

And hence,

$$H(\hat{s}) = \frac{1}{10\pi^2 \left(\hat{s}^2 + 0.1\hat{s} + 1\right)}$$

b). We will use a single pole controller, which has the form

$$K(\hat{s}) = \frac{K_o}{1 + 0.1\hat{s}}$$

The loop transfer function is hence,

$$H(\hat{s})K(\hat{s}) = \frac{K_o/10\pi^2}{(0.1\hat{s} + 1)\left(\hat{s}^2 + 0.1\hat{s} + 1\right)}$$

Using Matlab command rlocus(sys), the root locus plot of the closed loop transfer function is shown in Figure 1 below. A zoomed-in view of the root near the imaginary axis as shown Figure 2 reveals that the maximum gain K_o at which the two poles will coincide with the imaginary axis and hence the system will become unstable is 98.9. The Matlab code used to generate these two plots is provided.

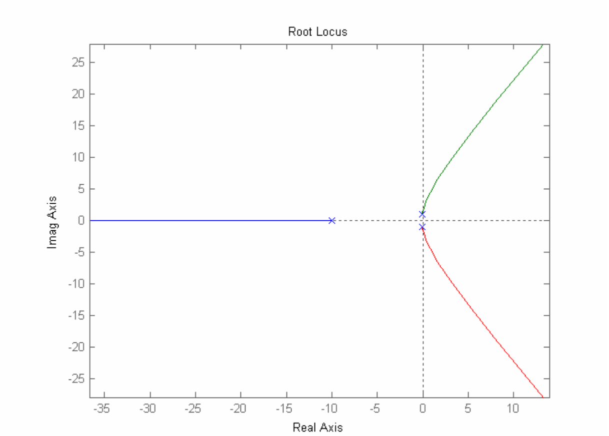


Figure 1. Root locus plot of the closed loop transfer function.

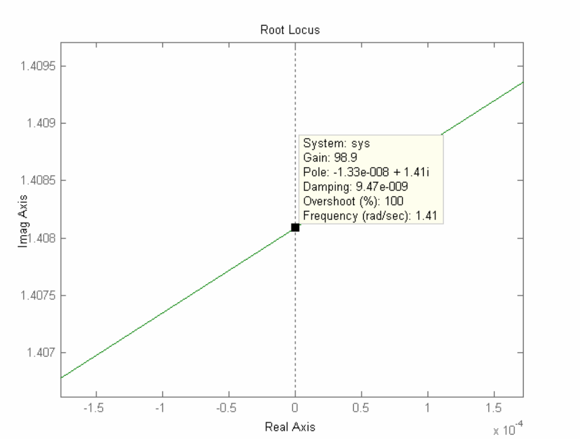


Figure 2. Zoomed-in view near the imaginary axis.

```
% root locus plot

clear all;
close all;

Hden=[1 0.1 1];
Hnum=[1/10/pi^2];
H=tf(Hnum,Hden)

Kden=[0.1 1];
Knum=[1];
K=tf(Knum,Kden);

sys=H*K;

rlocus(sys);
```

c). Instead of using a single-pole controller, we will design a PID controller that achieves overall critically damped system. A normalized PID controller has the form,

$$K(\hat{s}) = K_o\left(1+\frac{\beta}{\hat{s}}+\gamma\hat{s}\right)$$

The closed loop transfer function now becomes,

$$\frac{X_{out}}{X_{in}} = \frac{\frac{K_o}{k}\left(\gamma\hat{s}^2+\hat{s}+\beta\right)}{\hat{s}^3+\left(\frac{1}{Q}+\frac{K_o\gamma}{k}\right)\hat{s}^2+\left(1+\frac{K_o}{k}\right)\hat{s}+\frac{K_o\beta}{k}}$$

This is a third order system with 3 poles and 2 zeros. We know that for a second order system, critically damped response means that the system transfer function has two real poles that are equal, and as a result, the system achieves steady state with the fastest response without overshoot. In order for the system to exhibit a second order behavior, we ideally would want to have 2 equal real poles and a third pole that would cancel out a zero. It turns out that for this problem, it is not possible to achieve the critical damping with a PI controller. What we can do, however, is to have a third pole that is much larger than the 2 real poles such that its fast response does not have much noticeable effect on the second order system behavior. Therefore, we can express the denominator as,

$$\left(\hat{s}+a\right)\left(\hat{s}+b\right)^2, \textit{ where } a \gg b$$

And by comparison, we have,

$$\begin{cases} \frac{1}{Q}+\frac{K_o\gamma}{k} = a+2b \\ 1+\frac{K_o}{k} = b^2+2ab \\ \frac{K_o}{k}\beta = ab^2 \end{cases}$$

There could be many choices for a and b that could satisfy the criteria of critical damping, and without further specification on the rise time, for example, we will choose one pair that works. From part a), we can derive that the plant $H(s)$ has two complex poles $0.05 \pm j$. In order to have faster response, we want to have two real poles move to the left of the S-plane. Let's choose b=1, and a=1000, say. We can derive that,

$$\begin{cases} K_o = 2000k = 2\times10^4\pi^2 \\ \beta = 0.5 \\ \gamma = 0.501 \end{cases}$$

Hence the PID controller that is chosen is,

$$K(\hat{s}) = 2\times10^4\pi^2\left(1+\frac{0.5}{\hat{s}}+0.501\hat{s}\right)$$

The root locus plot of the new loop transfer function

$$H(\hat{s})K(\hat{s}) = \frac{2\times10^3\left(1+\frac{0.5}{\hat{s}}+0.501\hat{s}\right)}{\left(\hat{s}^2+0.1\hat{s}+1\right)}$$

is shown in Figure 3. The two zeros are very close to the real poles. Although it is advantageous to have the zeros far away from the dominant poles, since the zeros of a system affect pretty much the amplitude, rather than the oscillation nature of the system (as long as they are in the negative half of the s-plane), we do not have to worry about them too much. The step response of the overall closed loop system is shown in Figure 4, demonstrating the critical damped response, with a rise time on the order of 3 ms.

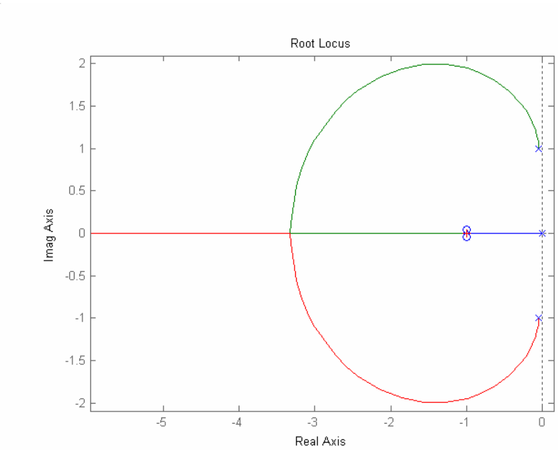


Figure 3. Root locus plot of the system with PID controller.

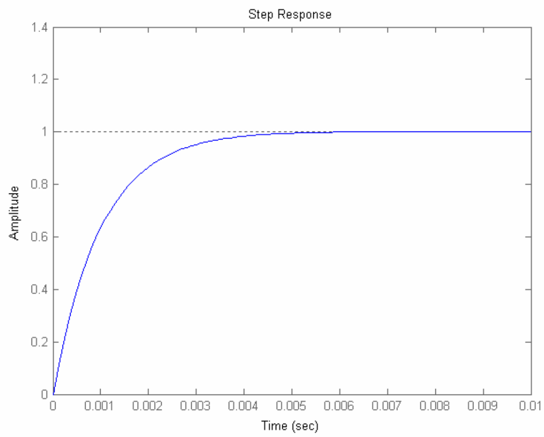


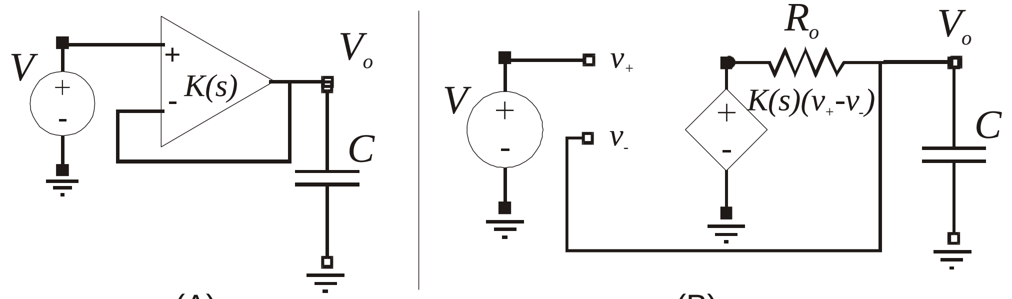
Figure 4. Step response of the system wit PID controller.

The MATLAB coding used for this portion of the problem is provided below:

```
% root locus plot

clear all;
close all;

Hden=[1 0.1 1];
Hnum=[1/10/pi^2];
H=tf(Hnum,Hden)

Kden=[1 0];
ko=20000*pi^2;
Knum=ko*[0.501 1 0.5];
K=tf(Knum,Kden);

sys=H*K

rlocus(sys);

figure
step(sys/(1+sys),0.01)
```

Problem 15.8 (4pts): Stability and capacitive loads

a). Determine the loop transmission function H(s)K(s)



(A) (B)

From the circuit model on the left, we have

$$V_- = V_0$$
$$V_+ = V$$

Applying KCL at the node connecting the output resistor and the capacitor:

$$\frac{V_0 - K_s(V - V_0)}{R_0} + \frac{V_0}{\frac{1}{sC}} = 0$$

Rearrange terms, we get,

$$V_o = \frac{K(s)}{1 + R_o \cdot s \cdot C} \cdot (V - V_0)$$

The model is equivalent to a linear feedback system with an overall transfer function as,

$$V_0 = H(s)K(s)(V - V_0)$$

which is referred to as Black's formula.

The Loop transmission function is hence,

$$H(s)K(s) = \frac{K(s)}{1 + R_o \cdot s \cdot C}$$

b). Assuming

$$K(s) = \frac{10^{3.5}}{\left(1 + {s}/{2\pi \times 10^5}\right)\left(1 + {s}/{2\pi \times 10^8}\right)}$$

The transmission function then becomes

$$H(s)K(s) = \frac{10^{3.5}}{\left(1 + {s}/{2\pi \times 10^5}\right)\left(1 + {s}/{2\pi \times 10^8}\right)} \cdot \frac{1}{1 + R_o \cdot s \cdot C}$$

$$= \frac{3162.28\pi^2}{\pi^2 + s\left(0.5005\pi \times 10^{-5} + R_o \cdot C \cdot \pi^2\right) + s^2\left(0.25 \times 10^{-13} + 0.5005 \times 10^{-5} \cdot R_o \cdot C \cdot \pi\right) + s^3\left(0.25 \times 10^{-13} \cdot R_o \cdot C\right)}$$

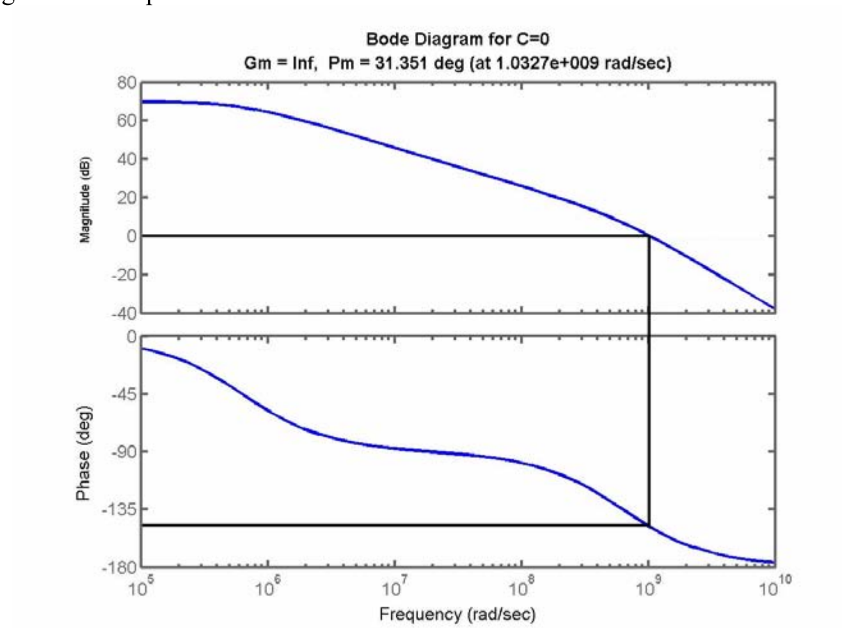
For the case C = 0, the loop transmission function becomes

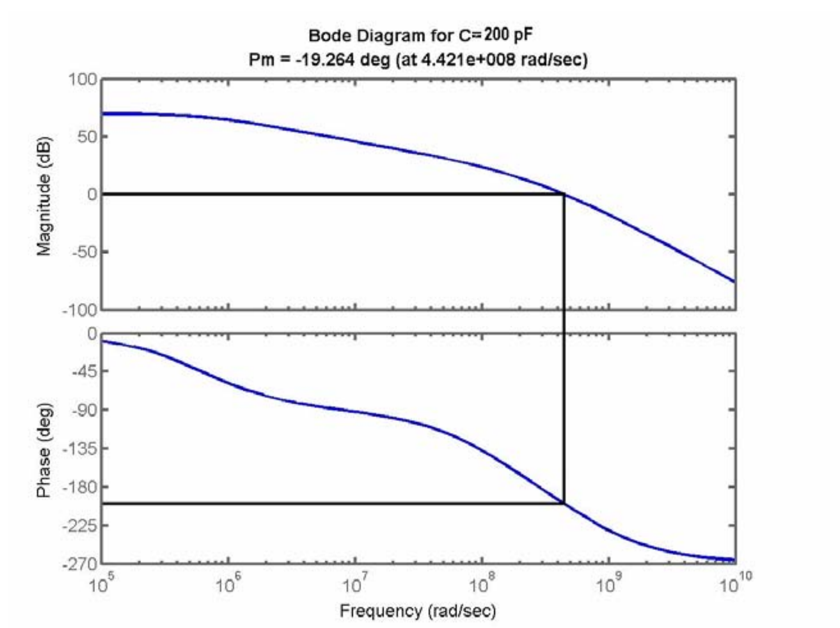
$$H(s)K(s) = \frac{3162.28\pi^2}{\pi^2 + s \cdot \left(0.5005\pi \times 10^{-5}\right) + s^2\left(0.25 \times 10^{-13}\right)}$$

For the case C = $200\text{x}10^{-9}$ F, the loop transmission function becomes

$$H(s)K(s) = \frac{3162.28 \cdot \pi^2}{\pi^2 + s\left(1.58 \times 10^{-5}\right) + s^2\left(1.5079 \times 10^{-13}\right) + s^3\left(2 \times 10^{-22}\right)}$$

The phase margin and bode plots of the two cases are shown below.





The phase margin angle γ is defined as

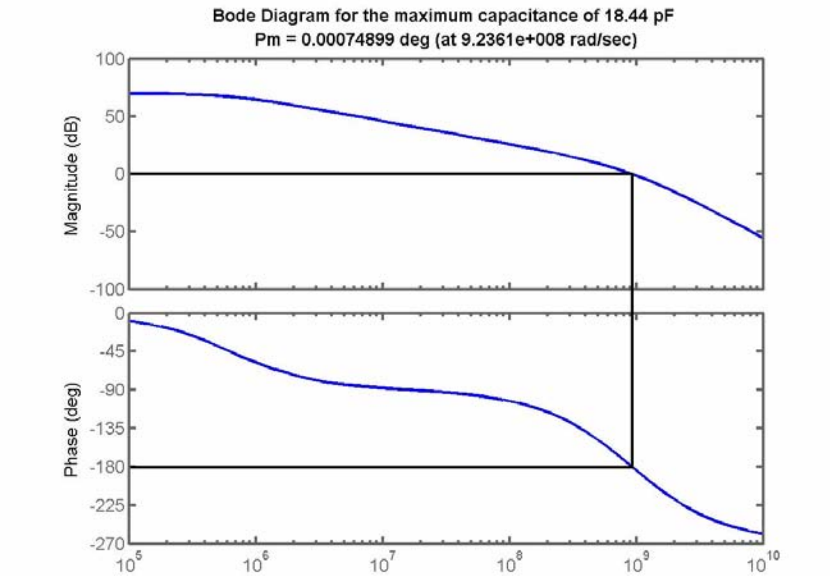
$$\gamma = 180^{o} + \alpha$$

where α is the phase angle where the amplitude of the output signal is equal to the amplitude of the input signal. A system is stable implies a positive phase margin value.

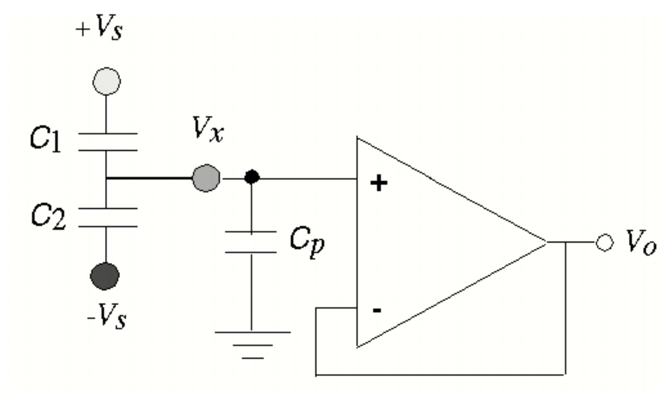
For the capacitance value of 0 F, the phase margin is 31.351^0.

For the case with C = 200 pF = $200\text{x}10^{-12}$ F the phase margin is -19.264^0. The system is unstable. We can see that a larger capacitance implies a smaller phase margin.

The phase margin is zero when the capacitance is decreased to about 18.44 pF, which is the maximum capacitance to be driven stably and the corresponding frequency is 147 MHz.



Problem 16.6 (10 pts): **Noise in a (vastly simplified) capacitive accelerometer**



a). In general, the force-displacement characteristic for a spring-mass-damper system is described as follows:

Knowing that the mass of the accelerometer is m=300 ng, and a resonant frequency $\omega_o = 2\pi \times 25000$ rad/s, and they system is critically damped, we can expressed the system transfer function for position x as,

$$H(s) = \frac{x(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

$$= \frac{1/m}{s^2 + \left(b/m\right)s + k/m}$$

In our system, we know that we are in a critically damped situation. We can there for insert the following expression in the generalized transfer function expression.

$$\frac{m\omega_0}{b} = \frac{1}{2} = Q \text{ for the critically damped case}$$

$$\text{where } \omega_0 = \sqrt{\frac{k}{m}}$$

This substitution yields the following simplified relation:

$$\begin{aligned} H(s) &= \frac{x(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \\ &= \frac{1/m}{s^2 + 2\omega_0 s + \omega_0^2} \\ &= \frac{1/m}{\left(s + \omega_0\right)^2} \end{aligned}$$

The intrinsic noise due to damping is similar to the noise generated by a resistor, and the spectral density function of this noise has the form,

$$S_n(f) = 4k_B Tb$$

where b=2$m\,\omega_o$ for the case of critical damping.

Though we could solve this problem using the following integral:

$$x_n^2 = \int_0^{\Delta f} \left| H\left(j2\pi f \right) \right|^2 S_n\left(f \right) df$$

.

we can instead approach to the problem by using the fact that the system bandwidth is much smaller than the resonant frequency of the accelerometer. Hence, we can approximate the quasi-static response, and use the bandwidth as the noise bandwidth. The mean square force acting on the mass is,

$$\overline{f_n^2} = 4k_B Tb\Delta f$$

So the displacement is,

$$x_n = \frac{f_n}{k} = \frac{\sqrt{4k_B Tb\Delta f}}{m\omega_o^2} = 5.337 \ pm$$

which is nearly the same as obtained by performing the integral

b). The effective capacitance C_1 and C_2 can be expressed as

$$\begin{aligned} C_{sense} &= \frac{\varepsilon_0 H L_0}{G_0 \pm x_{n,rms}} \\ &= \frac{\varepsilon_0 H L_0}{G_0} \cdot \frac{1}{1 \pm \frac{x_{n,rms}}{G_0}} \\ &= C_0 \frac{1}{1 \pm \frac{x_{n,rms}}{G_0}} \end{aligned}$$

where C_0 is the capacitance at zero displacement, and G_0 is the initial gap. We can further simplify this relation by linearizing around the initial gap condition using Taylor series expansion where we retain only the first term. We outline such a procedure here:

$$f = C_0 \frac{1}{1 \pm \frac{x_{n,rms}}{G_0}} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{-C_0}{G_0\left(1 \pm \frac{x_{n,rms}}{G_0}\right)^2}$$

$$\left.\frac{\partial f}{\partial y}\right|_{G_0} = \frac{-C_0}{G_0\left(1 \pm \frac{0}{G_0}\right)^2} = \frac{-C_0}{G_0}$$

The linearized form when retaining only the 1st Taylor series term:

$$C_{\text{sense}} \approx f\big|_{G_0} + \left(\left.\frac{\partial f}{\partial y}\right|_{G_0}\right) x_{n,rms} = C_0 + \left(\left.\frac{\partial f}{\partial y}\right|_{G_0}\right) x_{n,rms} = C_0\left(1 \mp \frac{x_{n,rms}}{G_0}\right)$$

The effective capacitance due to the fluctuation of position is therefore:

$$C_{sense} \approx 100\times10^{-15} \times (1 \pm \frac{5.337\times10^{-12}}{1.3\times10^{-6}})$$

The effective variation in C_1 and C_2 due to the fluctuation is $4.105\times10^{-19} F$.

To find the relation between Vs and V_x, apply KCL at the node connecting C_1 and C_2,

$$\frac{V_s - V_x}{\frac{1}{sC_1}} - \frac{V_x - (-V_s)}{\frac{1}{sC_2}} - \frac{V_x}{\frac{1}{sCp}} = 0$$

Simplify the equation, we have

$$V_x = \frac{C_1 - C_2}{C_1 + C_2 + C_p} V_s$$

We reasonably assume that with 2.5 V actuation, the nominal capacitances are relatively unchanged. Hence, the effective voltage noise source at the point of V_x is

$$v_T = \frac{C_0\left(1 + \frac{x_{n,rms}}{G_0}\right) - C_0\left(1 - \frac{x_{n,rms}}{G_0}\right)}{3C_0} \cdot V_s = \frac{2\frac{x_{n,rms}}{G_0}}{3} \cdot V_s$$
$$= \frac{2\cdot\frac{5.337\times10^{-12}}{1.3\times10^{-6}}}{3} \cdot 2.5$$
$$= 6.842\times10^{-6} V$$

c). The mean square voltage noise on the capacitors in our system is described using equation 16.23 from the text. We repeat it here in a slightly altered form.

$$v_{C,rms} = v_{C1} = v_{C2} = v_{Cp} = \sqrt{4k_B TR\Delta f}$$

In this expression, R represents the resistor that connects to the capacitors and also the thermal reservoir. Δf is the band width of interest. For the architecture outlined in this problem, we calculate the resistance using the following routine:

$$R = \rho \frac{L}{nA}$$
$$= 400\times10^{-6} \times 10^{-2} \Omega\cdot m \times \frac{100\times10^{-6} m}{25\times(25\times10^{-12} m^2)}$$
$$= 0.64\,\Omega$$

Pluging in, the root mean square voltage noise on the capacitors is:

$$\begin{aligned} v_{C,rms} &= v_{C1} = v_{C2} = v_{Cp} = \sqrt{4k_B TR\Delta f} \\ &= \sqrt{4\times 1.38\times 10^{-23}\, J/K \times 300K \times 0.64\Omega \times 1000Hz} \\ &= 3.256\times 10^{-9} V \end{aligned}$$

d). The mean square noise at the two inputs of the operational amplifier can be written using equation 16.19.

$$\overline{v_A^2} = \overline{v_B^2} = \int_{f_{cut-off}}^{1000} S_n(f)df$$

where, from equation 16.32

$$S_n(f) = 4k_B T(\frac{2}{3g_m})(1+F_n) + \frac{K_f}{WL\hat{C}_{0x} f}$$

In this expression:

g_m is the transistor transconductance
F_n is the noise factor (2~5), assuming 5
K_f is a scale factor for the 1/f noise, which is 10^{-24} V^2 F
$WL\hat{C}_{0x}$ is the gate-to-channel capacitance and
$\hat{C}_{0x} = \frac{\varepsilon_{0x}}{t_{ox}}$
ε_{ox} is the permittivity of oxide which is $3.9\varepsilon_0$
t_{ox} is the gate oxide thickness

Substituting in gives:

$$S_n(f) = 4\times 1.38\times 10^{-23}\times 300\times \frac{2}{3\times 300\times 10^{-6}}\times(1+5) + \frac{10^{-24}}{30\times 2\times 10^{-12}\times \frac{3.9\times 8.85\times 10^{-12}}{15\times 10^{-9}} f}$$

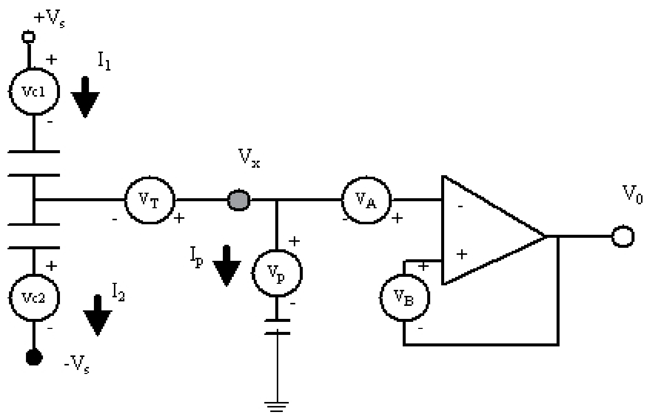
$$= 2.2080\times 10^{-16} + \frac{7.2432\times 10^{-12}}{f} V^2/Hz$$

The mean-square voltage noise

$$\begin{aligned} \overline{v_A^2} &= \overline{v_B^2} = \int_{f_{cut-off}}^{1000} S_n(f)df \\ &= \int_{10}^{1000} (2.2080\times 10^{-16} + \frac{7.2432\times 10^{-12}}{f})df \\ &= 3.3649\times 10^{-11} V^2 \end{aligned}$$

$$v_{A,rms} = v_{B,rms} = 5.008\times 10^{-6} V$$

e). The equivalent circuit is shown below.



$$I_1 = \frac{(V_s - v_{c1}) - (V_x - v_T)}{\frac{1}{sC_1}} \qquad I_2 = \frac{(V_x - v_T) - (-V_s + v_{c2})}{\frac{1}{sC_2}} \qquad I_p = \frac{V_x - v_p}{\frac{1}{sC_p}}$$

Apply KCL:

$$I_1 = I_2 + I_p$$

Substitute in

$$\frac{(V_s - v_{c1}) - (V_x - v_T)}{\frac{1}{sC_1}} = \frac{(V_x - v_T) - (-V_s + v_{c2})}{\frac{1}{sC_2}} + \frac{V_x - v_p}{\frac{1}{sC_p}}$$

Simplify the equation, we have

$$V_x = \frac{(V_s + v_T - v_{c1})C_1 + v_p C_p - (V_s - v_{C2} - v_T)C_2}{C_1 + C_2 + C_p}$$

where V_s is the signal. Since the noise sources are not correlated, the equivalent mean square noise at V_x is

$$\overline{V_{n,x}^2} = \left(\frac{C_1}{C_1 + C_2 + C_p}\right)^2 (\overline{v_T^2} + \overline{v_{c1}^2}) + \left(\frac{C_2}{C_1 + C_2 + C_p}\right)^2 \left(\overline{v_T^2} + \overline{v_{c2}^2}\right) + \left(\frac{C_p}{C_1 + C_2 + C_p}\right)^2 \overline{v_p^2}$$

in accordance with the remark at the bottom of p. 434 in the text.

Using the short method for the op-amp circuit, we have

$$V_0 - v_B = V_x + v_A$$
$$V_0 = v_B + v_A + V_x$$

The three noise sources are again uncorrelated, the mean square noise at the output is

$$\overline{V_{n,0}^2} = \overline{V_{n,x}^2} + \overline{v_B^2} + \overline{v_A^2}$$

Substitute in $\overline{V_{n,x}^2}$ and $C_1 = C_2 = C_p$, we have

$$\begin{aligned}
\overline{V_{n,0}^2} &= \frac{2}{9}\overline{v_T^2} + \frac{1}{9}\overline{v_{c1}^2} + \frac{1}{9}\overline{v_{c2}^2} + \frac{1}{9}\overline{v_{cp}^2} + \overline{v_B^2} + \overline{v_A^2} \\
&= \frac{2}{9}(6.842\times10^{-6})^2 + \frac{1}{3}(3.256\times10^{-9})^2 + 2\times(5.008\times10^{-6})^2 \\
&= 6.0563\times10^{-11} V^2
\end{aligned}$$

The root mean square of the noise at the output is

$$v_{n,o,rms} = 7.7822\times10^{-6}V$$

The total noise output is seen to be dominant by the thermal noise and the amplifier noise. These two noise sources have same order of magnitude and are three orders of magnitude larger than the capacitive noises. The transfer function of this circuit does not change the proportion of the contribution of the noise sources and hence we conclude that the form of the transfer function has no significant affect on the dominant terms of the noise.