Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.685 Electric Machines

Single Phase Induction Motors, Modeling of Inductances and Resistances

1 Introduction

This short note is a description of the analysis of single phase induction motors. It is intended to be read along with a description of the analysis of polyphase motors[1].This will be called the 'reference document'. The style of analysis used here is described in more detail in that note and many of the elements of this analysis appear only in that other note.

The single phase machine contemplated here actually has two windings, usually referred to as the "main" or "run" winding and the "auxiliary" or "start" winding. We assume here that the two windings are geometrically in quadrature but they need not have the same number of turns or the same distribution. A cartoon view of an axial section of such a machine is shown in Figure 1. We assume the direction of rotation is counter-clockwise (in the θ direction). The two windings are labeled as A (run) and B(start). The usual strategy for making one of these motors start is to arrange the impedances of the two windings so that current flows in the start winding with a phase *advance* with respect to the current in the run winding.

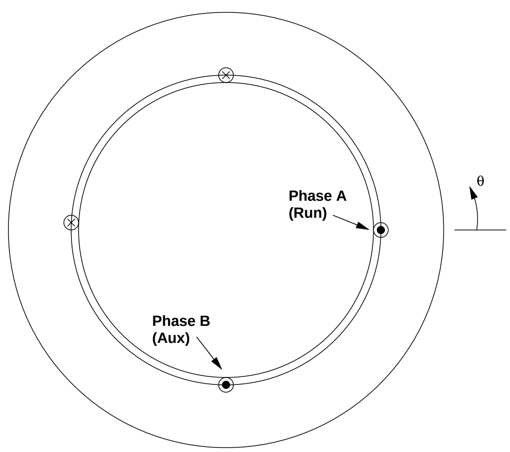

Figure 1: Single Phase Motor: Geometric Cartoon

2 Derivation of Fields and Inductances

To get started, we find the fields produced by the stator windings and their flux linkages so as to find their self inductances. In fact we will find only the air-gap inductances as these are all that

[1]For example, see "Analytic Design Evaluation of Induction Machines", Notes for MIT subject 6.685, Chapter 8. Available through MIT Open Courseware

are required for this explication. Of course analytical evaluation of machine operation will require evaluation of leakage inductances too.

With the geometry shown in Figure 1, radial magnetic flux density in the air-gap is:

$$B_r = \sum_{\text{n odd}} \mu_0 \frac{4}{n\pi} \left(\frac{N_a i_a}{2pg} k_{wna} \sin np\theta + \frac{N_b i_b}{2pg} k_{wnb} \sin n\frac{\pi}{2} \cos np\theta \right)$$

where N_a is the number of turns in the main winding, N_b is the number of turns in the auxiliary winding, p is the number of pole pairs, g is the effective air-gap and k_{wna} and k_{wnb} are the winding factors for the n^{th} harmonic fields of the two windings, respectively. Winding factors will be dealt with later.

To estimate inductance we need to compute flux linkages of the two windings. The flux linkages for the n^{th} harmonic fields are:

$$\begin{aligned} \lambda_{an} &= \ell \int_{-\frac{\pi}{p}}^{0} N_a k_{wna} B_{rn}(\theta) R d\theta \\ \lambda_{bn} &= \ell \int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}} N_b k_{wnb} B_{rn}(\theta) R d\theta \end{aligned}$$

Here, we have used the symbol ℓ to describe the axial length of the machine.

To estimate winding self inductance, in which we ignore any currents that might be flowing in the rotor, we find the n^{th} space harmonic flux linkages to be (no surprise here):

$$\begin{aligned} \lambda_{an} &= \mu_0 \frac{4}{\pi} \frac{N_a^2 k_{wna}^2 R\ell}{n^2 p^2 g} i_a \\ \lambda_{bn} &= \mu_0 \frac{4}{\pi} \frac{N_b^2 k_{wnb}^2 R\ell}{n^2 p^2 g} i_b \end{aligned}$$

Where R is radius of the air-gap.

At this point we define the effective turns ratio

$$\alpha_n = \sin n\frac{\pi}{2} \frac{N_b k_{wnb}}{N_a k_{wna}}$$

so that the n^{th} harmonic inductances of the windings are:

$$\begin{aligned} L_{an} &= \mu_0 \frac{4}{\pi} \frac{N_a^2 k_{wna}^2 R\ell}{n^2 p^2 g} \\ L_{bn} &= \mu_0 \frac{4}{\pi} \frac{N_b^2 k_{wnb}^2 R\ell}{n^2 p^2 g} = \alpha_n^2 L_{an} \end{aligned}$$

Note the sign of the effective turns ratio, which does not affect the self inductances but does affect the direction of rotation of the various magnetic field components. Harmonics number 3 and 7 (and one of the zigzag components) rotate in the reverse direction than would be indicated by the sequence order of currents in the stator windings.

3 Squirrel-Cage Model

In the single phase motor we must deal with a number of space harmonics with positive and negative phase velocities. This analytical description will deal with one such component. The squirrel cage is made up of a number (N_R) of discrete conductors ('bars'), each carrying a current that will be of the form (in bar k):

$$i_k = \text{Re}\left\{\underline{I}_k e^{j\omega_r t}\right\}$$

Depending on the direction of the current wave the bar current will be of the form:

$$\underline{I}_k = \underline{I}_0 e^{\mp j\frac{2\pi npk}{N_R}}$$

As is shown in the reference document, this set of discrete currents is equivalent to a number of space harmonics of surface current:

$$K_z = \text{Re}\left\{\sum_n \frac{N_R I_0}{2\pi R} e^{j(\omega_r t \mp np\theta')}\right\}$$

The values of n for which K_z is non-zero are

$$n = 1 + \text{integer} \times \frac{N_R}{p}$$

and they will produce magnetic flux across the air-gap:

$$\underline{B}_{rn} = \mp j\mu_0 \frac{N_R I_0}{2\pi npg}$$

Electric field induced by these fields is:

$$\underline{E}_n = \pm\frac{\omega_r R}{np}\underline{B}_n = -j\frac{\mu_0 N_R \omega_r R}{2\pi g n^2 p^2}\underline{I}_0$$

If we assume, as in the reference document, that only the smallest order terms ($n = 1$, $n = 1 \pm \frac{N_R}{p}$) contribute substantially to the electric field driving current through the rotor conductor,

$$\underline{E}_1 + \underline{E}_{n+} + \underline{E}_{n-} = Z_{\text{slot}}\underline{I}_0$$

Now we need to put this back into the form of an equivalent circuit. Note that the space fundamental field is caused by currents in the stator as well as the rotor but that the higher order space harmonic voltage components are produced only by rotor currents. To refer the rotor current back to the stator, note that, if a current I_F were in the stator it would make a magnetic field:

$$\underline{B}_r = -\mu_0 \frac{4}{\pi}\frac{N_a \underline{I}_F k_{wna}}{2pg}$$

then the correct turns ratio to refer rotor current to the stator is:

$$\underline{I}_0 = \frac{4N_a k_{wna}}{N_R}\underline{I}_F$$

So now the space fundamental component of electric field seen from the rotor is:

$$\underline{E}_1 = \left(\frac{4N_a k_{w1a}}{N_R} + j\omega_r \frac{2}{\pi} \frac{\mu_0 N_a k_{wna} R}{g} \left(\frac{1}{(N_R+p)^2} + \frac{1}{(N_R-p)^2} \right) \right) \underline{I}_F$$

Finally, to relate stator air-gap voltage to rotor electric field, we must do two things: first, translate electric field according to relative frequencies between rotor and stator and integrate over the length of the winding. The result is:

$$\underline{V}_{ag} = -2\ell N_a k_{w1a} \frac{\omega}{\omega_r} \underline{E}_1$$

The equivalent circuit elements for the rotor are then just:

$$R_2 = \frac{8\ell N_a^2 k_{w1a}^2}{N_R} R_{\text{slot}}$$

$$X_2 = \frac{8\ell N_a^2 k_{w1a}^2}{N_R} \omega L_{\text{slot}} + \omega \frac{4}{\pi} \frac{\mu_0 N_a^2 k_{w1a}^2 R\ell}{g} \left(\frac{1}{(N_R+p)^2} + \frac{1}{(N_R-p)^2} \right)$$

Note that the slot resistance and reactance parameters may be frequency dependent and some care must be taken to compute those parameters at the right frequency. Note also that the same extension to space harmonics used for polyphase machines in the reference document will be useful here. Note, however, that the triplen harmonics will, in general, be present and important in the single phase machine, unlike three phase motors. Thus the magnetizing inductance, slot leakage and rotor resistance for the higher order harmonic terms will be of the form:

$$L_{agn} = \frac{4}{\pi} \frac{\mu_0 N_a^2 k_{wan}^2 R\ell}{n^2 p^2 g}$$

$$X_{2,n} = \frac{8\ell N_a^2 k_{wna}^2}{N_R} \omega L_{\text{xlot}} + \omega \frac{4}{\pi} \frac{\mu_0 N_a^2 k_{wna}^2 R\ell}{g} \left(\frac{1}{(N_R+np)^2} + \frac{1}{(N_R-np)^2} \right)$$

$$R_{2,n} = \frac{8\ell N_a^2 k_{wna}^2}{N_R} R_{\text{slot}}$$

Note that rotor resistance must be corrected for end ring effects as described in the reference document.

4 Winding Factor

In the single phase machines the windings can be described as 'concentric', or as a collection of some number of coils, all with the same axis, with different coil throws and perhaps a different number of turns. If we denote $N_s(k)$ as the number of turns in coil k and $N_c(k)$ as the coil throw, then the total number of turns is just the sum of all of the N_s's and the electrical span angle for coil k is

$$\phi_k = \frac{2p\pi N_c(k)}{S}$$

where S is the total number of slots in the stator (this assumes the slots have equal spacing, which may be a limitation here). The winding factor is then the weighted sum of the winding factors of all of the coils:

$$
\begin{aligned}
N_a &= \sum_k N_s(k) \\
k_{wna} &= \sum_k \frac{N_s(k)}{N_a} \sin \frac{n\phi_k}{2}
\end{aligned}
$$

5 Skew

Often, rotors are skewed, in which case the rotor and stator link different fluxes. The *self* inductance will be what is computed in this note but the *mutual* inductances will be modified by the skew factor. The difference beween self and mutual inductance must be treated as *leakage* inductance.

If the skew from one end of the rotor to the other is, in electrical degrees, θ_{sk}, the skew factor for the n^{th} harmonic can be shown to be:

$$
k_{sn} = \frac{\sin(\frac{n\theta_{sk}}{2})}{(\frac{n\theta_{sk}}{2})}
$$

The magnetizing and leakage inductance components are then:

$$
\begin{aligned}
X_{\phi n} &= \omega L_{agn} k_{sn}^2 \\
X_{1n} &= \omega L_{agn}(1 - k_{sn}^2)
\end{aligned}
$$

6 Operation: Fundamental only

Admitting that space harmonics may be important here, we outline operation considering the space fundamental only. Space harmonics can be added conveniently once the basic operation is understood.

The coordinate system shown in Figure 2 is used.

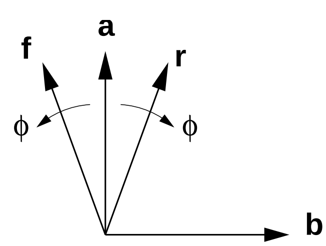

Figure 2: Rotating Field Coordinates

If we assume that the two stator phases are noted as a and b and two equivalent rotor phases are A and B, flux linkages are:

$$
\begin{aligned}
\lambda_a &= L_a i_a + L_\phi i_A \\
\lambda_b &= L_b i_b + \alpha L_\phi i_B \\
\lambda_A &= L_\phi i_a + L_A i_A \\
\lambda_B &= \alpha L_\phi i_b + L_A i_B
\end{aligned}
$$

where we have assumed that the rotor equivalent winding has the same number of turns and winding factor as the run winding of the stator.

Now this is not a convenient set to use because the interaction of the rotor makes the equivalent phases A and B difficult to use. So we will formulate a coordinate transformation. Working in complex amplitudes, we assume that the equivalent quantities in the rotor coordinates are the sum of components rotating forward and backward:

$$\begin{bmatrix} \underline{I}_A \\ \underline{I}_B \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} \underline{I}_F \\ \underline{I}_R \end{bmatrix}$$

The inverse transformation is:

$$\begin{bmatrix} \underline{I}_F \\ \underline{I}_R \end{bmatrix} = \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \begin{bmatrix} \underline{I}_A \\ \underline{I}_B \end{bmatrix}$$

Then the complex amplitudes of flux linkages are:

$$\begin{aligned}
\underline{\Lambda}_a &= L_a \underline{I}_a + \frac{L_\phi}{2} \underline{I}_F + \frac{L_\phi}{2} \underline{I}_R \\
\underline{\Lambda}_b &= L_b \underline{I}_b + \frac{j\alpha L_\phi}{2} \underline{I}_F - \frac{j\alpha L_\phi}{2} \underline{I}_R \\
\underline{\Lambda}_F &= L_\phi \underline{I}_a - j\alpha L_\phi \underline{I}_b + L_A \underline{I}_F \\
\underline{\Lambda}_R &= L_\phi \underline{I}_a + j\alpha L_\phi \underline{I}_b + L_A \underline{I}_R
\end{aligned}$$

Voltage equations are, in the stator coordinate system:

$$\begin{aligned}
\underline{V}_a &= (jX_a + R_a)\underline{I}_a + \frac{jX_\phi}{2} \underline{I}_F + \frac{jX_\phi}{2} \underline{I}_R \\
\underline{V}_b &= (jX_b + R_b)\underline{I}_b - \frac{\alpha X_\phi}{2} \underline{I}_F + \frac{\alpha X_\phi}{2} \underline{I}_R \\
0 &= \frac{jX_\phi}{2} \underline{I}_a + \frac{\alpha X_\phi}{2} \underline{I}_b + \left(\frac{jX_A}{2} + \frac{R_2}{2s} \right) \underline{I}_F \\
0 &= \frac{jX_\phi}{2} \underline{I}_a - \frac{\alpha X_\phi}{2} \underline{I}_b + \left(\frac{jX_A}{2} + \frac{R_2}{2(2-s)} \right) \underline{I}_R
\end{aligned}$$

This set of four linear equations is readily solved for the four currents I_a, I_b, I_F and I_R. To find mechanical energy converted, see that air-gap power and power dissipated on the rotor are, (working in RMS):

$$\begin{aligned}
P_{ag} &= |I_F|^2 \frac{R_2}{2s} + |I_R|^2 \frac{R_2}{2(2-s)} \\
P_d &= |I_F|^2 \frac{R_2}{2} + |I_R|^2 \frac{R_2}{2}
\end{aligned}$$

Mechanical energy converted is the difference:

$$P_m = |I_F|^2 \frac{R_2}{2s}(1-s) - |I_R|^2 \frac{R_2}{2(2-s)}(1-s)$$

and torque is

$$T_m = \frac{p}{\omega(1-s)} P_m = \frac{p}{\omega} \left[|I_F|^2 \frac{R_2}{2s} - |I_R|^2 \frac{R_2}{2(2-s)} \right]$$

7 Operation With Space Harmonics

The space harmonics couple together only in the stator winding which produces the space harmonic fields in response to armature currents. They are independent of each other in the rotor, however. The coupling is reflected in an addition of all of the harmonic components in the production of voltage in the stator. Each rotating component (forward and backward at each harmonic order) will have its own voltage balance equation. Considering only one of the harmonics, which we will refer to as n, the voltage equations become:

$$\begin{aligned}
\underline{V}_A &= (jX_a + R_a)\underline{I}_a + \frac{jX_\phi}{2}\underline{I}_F + \frac{jX_\phi}{2}\underline{I}_R + \frac{jX_{\phi n}}{2}\underline{I}_{Fn} + \frac{jX_{\phi n}}{2}\underline{I}_{Rn} \\
\underline{V}_B &= (jX_b + R_b)\underline{I}_b - \frac{\alpha X_\phi}{2}\underline{I}_F + \frac{\alpha X_\phi}{2}\underline{I}_R - \frac{\alpha_n X_{\phi n}}{2}\underline{I}_{Fn} + \frac{\alpha_n X_{\phi n}}{2}\underline{I}_{Rn} \\
0 &= \frac{jX_\phi}{2}\underline{I}_a + \frac{\alpha X_\phi}{2}\underline{I}_b + \left(\frac{jX_A}{2} + \frac{R_2}{2s}\right)\underline{I}_F \\
0 &= \frac{jX_\phi}{2}\underline{I}_a - \frac{\alpha X_\phi}{2}\underline{I}_b + \left(\frac{jX_A}{2} + \frac{R_2}{2(2-s)}\right)\underline{I}_R \\
0 &= \frac{jX_{\phi n}}{2}\underline{I}_a + \frac{\alpha_n X_{\phi n}}{2}\underline{I}_b + \left(\frac{jX_{An}}{2} + \frac{R_{2,n}}{2s_{n+}}\right)\underline{I}_{Fn} \\
0 &= \frac{jX_{\phi n}}{2}\underline{I}_a - \frac{\alpha_n X_{\phi n}}{2}\underline{I}_b + \left(\frac{jX_{An}}{2} + \frac{R_{2,n}}{2s_{n-}}\right)\underline{I}_{Rn}
\end{aligned}$$

Harmonic slips are the ratio between rotor frequency and stator frequency and are:

$$\begin{aligned}
s_{n+} &= ns - (n-1) \\
s_{n-} &= (n+1) - ns
\end{aligned}$$

Torques from the space harmonics are estimated in the same way as for the fundamental: air-gap and dissipated power are:

$$\begin{aligned}
P_{ag} &= |I_{Fn}|^2\frac{R_2}{2s_{n+}} + |I_{Rn}|^2\frac{R_2}{2s_{n-}} \\
P_d &= |I_{Fn}|^2\frac{R_2}{2} + |I_{Rn}|^2\frac{R_2}{2}
\end{aligned}$$

Inserting the definition for harmonic order slip, we find torque due to the n^{th} harmonic is:

$$T_{mn} = \frac{np}{\omega}\left(|I_{Fn}|^2\frac{R_2}{2s_{n+}} - |I_{Rn}|^2\frac{R_2}{2s_{n-}}\right)$$

Adding harmonic terms is straightforward and though, if there are a lot of space harmonics considered, it yields a relatively large coupling matrix, the resulting linear equation set is straightforward to solve.

Generally, in a single phase motor the auxiliary circuit is connected to an external impedance (e.g. a capacitor or parallel combination of a capacitor and a resistor) and then to the same voltage source as the main winding. The resulting set of expressions is then:

$$
\begin{aligned}
\underline{V} &= (jX_a + R_a)\underline{I}_a + \frac{jX_\phi}{2}\underline{I}_F + \frac{jX_\phi}{2}\underline{I}_R + \frac{jX_{\phi n}}{2}\underline{I}_{Fn} + \frac{jX_{\phi n}}{2}\underline{I}_{Rn} \\
\underline{V} &= (jX_b + R_b + Z_e)\underline{I}_b - \frac{\alpha X_\phi}{2}\underline{I}_F + \frac{\alpha X_\phi}{2}\underline{I}_R - \frac{\alpha_n X_{\phi n}}{2}\underline{I}_{Fn} + \frac{\alpha_n X_{\phi n}}{2}\underline{I}_{Rn} \\
0 &= \frac{jX_\phi}{2}\underline{I}_a + \frac{\alpha X_\phi}{2}\underline{I}_b + \left(\frac{jX_A}{2} + \frac{R_2}{2s}\right)\underline{I}_F \\
0 &= \frac{jX_\phi}{2}\underline{I}_a - \frac{\alpha X_\phi}{2}\underline{I}_b + \left(\frac{jX_A}{2} + \frac{R_2}{2(2-s)}\right)\underline{I}_R \\
0 &= \frac{jX_{\phi n}}{2}\underline{I}_a + \frac{\alpha_n X_{\phi n}}{2}\underline{I}_b + \left(\frac{jX_{An}}{2} + \frac{R_{2,n}}{2s_{n+}}\right)\underline{I}_{Fn} \\
0 &= \frac{jX_{\phi n}}{2}\underline{I}_a - \frac{\alpha_n X_{\phi n}}{2}\underline{I}_b + \left(\frac{jX_{An}}{2} + \frac{R_{2,n}}{2s_{n-}}\right)\underline{I}_{Rn}
\end{aligned}
$$

where Z_e is the value of the external impedance element.