Problem Set 1 Solutions September 6, 2013

Problem 1: If the coil is thin, Inductance is found easily to be:

$$L = \mu_0 \frac{N^2 \pi R^2}{h}$$

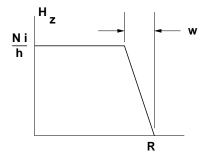If the coil is *not* thin, the magnetic field inside will be as shown in Figure 1.

Figure 1: Field Intensity vs. Radius

Flux linked by a coil filament at radius r is then:

$$\phi(r) = \mu_0 \frac{Ni}{h} \pi (R - w)^2 + \int_{R-w}^{r} \mu_0 \left(\frac{R - r'}{w} \right) 2\pi r' dr'$$

In principle, this could be evaluated and then averaged over the coil radius to get inductance. However, it appears (to this calculus duffer) to re-parameterize the problem by assigning $x = R - r$, the distance from the outside of the coil. Then:

$$
\begin{aligned}
H_z &= \frac{Ni}{h} \frac{x}{w} & x < W \\
&= \frac{Ni}{h} & x > w
\end{aligned}
$$

In which case, flux can be found for the region of the coil:

$$\phi(x) = \frac{\mu_0 Ni}{h} \left[\pi(R - w)^2 + \int_{x}^{w} 2\pi (R - x') \frac{x'}{w} dx' \right]$$

After some manipulation, this becomes:

$$\phi(x) = \frac{\mu_0 \pi Ni}{h} \left[R^2 - Rw + \frac{1}{3} w^2 - \frac{Rx^2}{w} + \frac{2}{3} \frac{x^3}{w} \right]$$

then inductance is found by averaging ϕ over the extent of the coil, and with a few steps of grunge left out:

$$
\begin{aligned}
L &= \frac{1}{i} \int_0^w \frac{N}{w} \phi(x) dx \\
&= \frac{\mu_0 \pi N^2}{hw} \int_0^W \left[R^2 - Rw + \frac{1}{3}w^2 - \frac{Rx^2}{w} + \frac{2}{3}\frac{x^3}{w} \right] dx \\
&= \frac{\mu_0 \pi N^2 R^2}{h} \left[1 - \frac{4}{3}\frac{w}{R} + \frac{1}{2}\left(\frac{w}{R}\right)^2 \right]
\end{aligned}
$$

Problem 2: With a steady DC current, field along the centerline is simply:

$$
H_0 = \frac{Ni}{h}
$$

In the event of a transient, Faraday's Law can be used:

$$
\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{n} da
$$

Taking this integral within the conductive cylinder at radius R_c, this is:

$$
\frac{2\pi R_c}{\sigma} (H_i - H_0) + \frac{d}{dt} \pi R_c^2 \mu_0 H_i = 0
$$

where H_i is the field inside and $H_0 = \frac{Ni}{h}$ is the field outside the shell.

The shell time constant is identified:

$$
T_s = \frac{\mu_0 \sigma t_s R_c}{2}
$$

and the differential equation for field is:
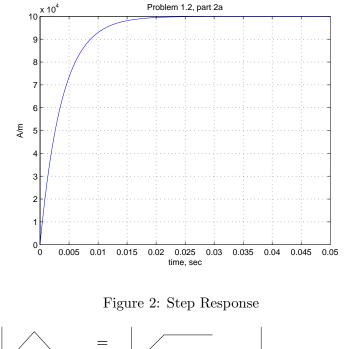
$$
T_s \frac{d}{dt} H_i + H_i = H_o
$$

Driven by a step of $H_o = \frac{Ni}{h} u(t)$,

$$
H_i = \frac{Ni}{h} \left(1 - e^{-\frac{t}{T_s}} \right)
$$

This is shown in Figure 2

To find the response of the field inside the shell, see that the symmetrical triangle wave can be expressed as the sum of two waveforms, each consisting of a ramp and a constant value, as shown in Figure 3. This observation allows us to calculate the magnetic field once and then add the same solution, inverted in sign and delayed.

First, we solve the field problem for the first excitation, consisting of a ramp of length T and a slopt of $\frac{NI}{hT}$, followed by a constant value of $\frac{NI}{h}$. During the ramp, since the ramp is the integral of the step, the response is the integral of the step response, and for the period after the end of the ramp, the solution follows a simple first-order response to the steady state field:
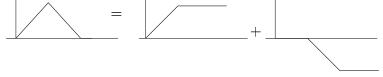
Figure 2: Step Response

Figure 3: Addition of two waves to form the triangle

$$H(t) = \qquad H_0\left[\frac{t}{\tau} - \frac{\tau}{T}\left(1 - e^{-\frac{t}{\tau}}\right)\right] \qquad\qquad 0 < t < T$$
$$H(t) = \quad H_0\left[1 - \frac{\tau}{T}\left(1 - e^{-\frac{T}{\tau}}\right)e^{-\frac{t-T}{\tau}}\right] \qquad\qquad t > T$$

It is straightforward to combine this solution with its inverted and shifted version, and now there are three time regions:

$$H(t) = \qquad\qquad\qquad H_0\left[\frac{t}{\tau} - \frac{\tau}{T}\left(1 - e^{-\frac{t}{\tau}}\right)\right] \qquad\qquad\qquad 0 < t < T$$
$$H(t) = \quad H_0\left[1 - \frac{\tau}{T}\left(1 - e^{-\frac{T}{\tau}}\right)e^{-\frac{t-T}{\tau}}\right] - H_0\left[\frac{t-T}{T} - \frac{\tau}{T}\left(1 - e^{-\frac{t-T}{\tau}}\right)\right] \qquad T < t < 2T$$
$$H(t) = \qquad\qquad\qquad H_0\frac{\tau}{T}\left(1 - e^{-\frac{T}{\tau}}\right)^2 e^{-\frac{t-2T}{\tau}} \qquad\qquad\qquad 2T < t$$

This is shown in Figure 4, along with the triangle wave that would be the field were the shell not there.

Finally (for this problem), we must find coil voltage. Since the coil has resistance and links time varying flux, we have:

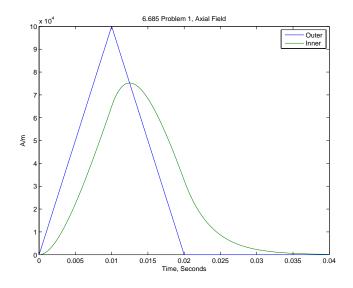$$v(t) = R_w I(t) + \frac{d\lambda}{dt}$$

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Figure 4: addition of two waves to form the triangle

Coil resistance is found by $R_w = \ell_w / \sigma A_w$. ℓ_w is total wire length, and it is found my multiplying average wire length by the number of turns:

$$\ell_w = 2\pi \left(R - \frac{w}{2} \right) N$$

Wire area A_w is found by multiplying the coil cross section by space factor and dividing by the number of turns:

$$A_w = \frac{hw\lambda_a}{N}$$

So coil resistance is

$$R_w = \frac{2\pi \left(R - \frac{w}{2} \right) N^2}{hw\lambda_a}$$

To find flux, we can observe that total flux linkage can be expressed as the sum of flux inside of the shell, which we have just calculated, plus flux that would be linked by the inductance that would exist were the region of the shell to be perfectly diamagnetic. That inductance is found by a minor modification of the calculation of inductance above:

$$L_0 = \frac{\mu_0 \pi N^2}{h} \left(R^2 - R_c^2 - \frac{4}{3}\frac{w}{R} + \frac{1}{2} \left(\frac{w}{R} \right)^2 \right)$$

To find the rate of change of flux within the shell, we could either differentiate the field we found above or take advantage of the fact that the response to the derivative of a function is the derivative of the response to that function, and note that the derivative of the exciting current is two steps. In either case, we get the derivative of the magnetic field inside the shell:

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$$\dot{H}(t) = H_d\left(1 - e^{-\frac{t}{\tau}}\right) \qquad 0 < t < T$$

$$\dot{H}(t) = H_d\left(1 - e^{-\frac{T}{\tau}}\right)e^{-\frac{t-T}{\tau}} - H_d\left(1 - e^{-\frac{t-T}{\tau}}\right) \qquad T < t < 2T$$

$$\dot{H}(t) = -H_d\left(1 - e^{-\frac{T}{\tau}}\right)^2 e^{-\frac{t-2T}{\tau}} \qquad 2T < t$$

where H_d is the slope of the magnetic field that would exist in the absence of the shell (the driving field):

$$H_d = \frac{NI}{hT}$$

With these fragments, we can calculate voltage:

$$v(t) = R_w i(t) + L_0 \frac{di}{dt} + \dot{\lambda}_s$$

where the derivative of shell linked flux is:

$$\dot{\lambda}_s(t) = \pi R_c^2 N \mu_0 \dot{H}(t)$$

The grungy calculations are carried out by a Matlab script that is appended. The three components of voltage are shown in Figure 5 and the final terminal voltage is shown in Figure 6.

For the dimensions given in the problem set, parameters turn out to be:

Resistance	40.8	Ω
Inductance L_0 (excluding the shell region	1.08	H
Inductance L (without the shell)	1.47	H
Approximate Inductance (ignoring coil thickness)	1.58	H

Problem 3 (Note this problem is mis-labeled in the original problem set as Problem 4)

Rotational speed is:
$$\Omega = 2\pi \times \frac{10,000}{60} \approx 1047 \text{Radians/sec}$$

and then torque is:
$$T = \frac{10,000}{1047} \approx 9.55\text{N-m}$$

Rotor volume is
$$V_R = \frac{T}{2\tau} = \frac{9.55}{10,000} = 4.77 \times 10^{-4}\text{m}^3$$

If rotor speed $R\Omega = 230$ m/s, $R = .22$ m. and then rotor length is:

$$\ell = \frac{4.77 \times 10^{-4}}{\pi \times .22^2} \approx .0031\text{m}$$

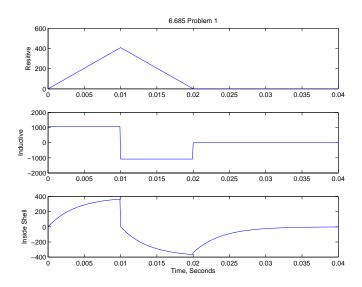This is a kind of strange machine! We would probably seek some other morphology...
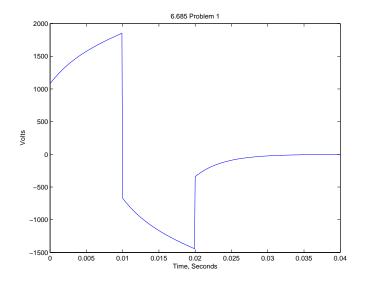
Figure 5: Components of Voltage

Figure 6: Coil Terminal Voltage

```matlab
% 6.685, Fall 2013 Problem Set 1, Problem 2

h = .1;                % height
R = .2;                % coil outer radius
w = .01;               % coil thickness
R_c = .1;              % shell radius
sig = 6e7;             % shell conductivity
ts = .001;             % shell thickness
muzero = 4e-7*pi;      % permeability of free space
N = 1000;              % turns in the coil
lama = 0.5;            % winding factor for the coil
tt = .01;              % transient time
tau = R_c*sig*ts*muzero/2;        % shell time constant
I_0 = 10;


fprintf('Problem 1, part 3 tau = %g\n', tau)


H_0 =N*I_0/h;          % this is the steady state field
Hdot_0 = H_0/tt;       % this is magnitude of outer field derivative


t = 0:.0001:.04;       % going to plot over this range of time


I = zeros(size(t));
Idot = zeros(size(t));
Ha = zeros(size(t));
H = zeros(size(t));
Hdot = zeros(size(t));


for k = 1:length(t);
    if t(k)<tt,
        Ha(k) = H_0 * t(k)/tt;
        H(k) = H_0 * (t(k)/tt - (tau/tt)*(1 - exp(-t(k)/tau)));
        Hdot(k) = Hdot_0 * (1 - exp(-t(k)/tau));
        I(k) = I_0 * t(k)/tt;
        Idot(k) = I_0/tt;
    elseif t(k)<2*tt
        Ha(k) = H_0*(1-(t(k)-tt)/tt);
        H(k) = H_0 * (1 - exp(-(t(k)-tt)/tau)*(tau/tt)*(1-exp(-tt/tau))...
            -((t(k)-tt)/tt - (tau/tt)*(1-exp(-(t(k)-tt)/tau))));
        Hdot(k) = Hdot_0 * ((1 - exp(-tt/tau)) * exp((-t(k)-tt)/tau)...
            -(1 - exp(-(t(k)-tt)/tau)));
        I(k) = I_0 * (1 - (t(k)-tt)/tt);
        Idot(k) = - I_0/tt;
    else
        Ha(k) = 0;
        H(k) = H_0*(tau/tt)*exp(-(t(k)-2*tt)/tau)*(1-exp(-tt/tau))^2;
        Hdot(k) = -Hdot_0 * (1 - exp(-tt/tau))^2 * exp(-(t(k)-2*tt)/tau);
```

```
        I(k) = 0;
        Idot(k) = 0;
    end
end

% so now plot magnetic field through the shield

figure(1)
plot(t, Ha, t, H)
title('6.685 Problem 1, Axial Field')
ylabel('A/m')
xlabel('Time, Seconds')
legend('Outer', 'Inner')

% Now get the parameters of the system

% winding resistance
Rw = 2*pi*(R-w/2)*N^2/(h*w*lama*sig);

% Inductance of outer part
L_0 = (muzero*pi*N^2/h) * (R^2 - R_c^2 - (4/3)*R*w + .5 * w^2);

fprintf('Winding Resistance = %g Ohms\n', Rw);
fprintf('Inductance (excluding inside shell = %g H\n', L_0)

L_t = (muzero*pi*N^2/h) * (R^2 - (4/3)*R*w + .5 * w^2);
L_a = (muzero*pi*N^2/h) * R^2;

fprintf('Inductance, total = %g\n', L_t);
fprintf('Inductance, Approximate = %g\n', L_a);

% voltage from flux linkage from inside the shell
Vr = Rw .* I;
Vl = L_0 .* Idot;
Vi = pi*R_c^2 * N * muzero .* Hdot;

V = Vr + Vl + Vi;

figure(2)
plot(t, Vr, t, Vl, t, Vi, t, V);
title('6.685 Problem 1, Voltages')
ylabel('Volts')
xlabel('Time, Seconds')
legend('Resistance', 'Inductance', 'Inside Shell', 'Total')

figure(3)
subplot 311
plot(t, Vr);
```

```
title('6.685 Problem 1')
ylabel('Resitive')
subplot 312
plot(t, Vl);
ylabel('Inductive')
subplot(313)
plot(t, Vi)
ylabel('Inside Shell')
xlabel('Time, Seconds')


figure(4)
clf
plot(t, V);
title('6.685 Problem 1')
ylabel('Volts')
xlabel('Time, Seconds')
```

6.685 Electric Machines
Fall 2013