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6.642 Continuum Electromechanics

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Linearized Surface Tension Force Density in Spherical Coordinates

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$$\mathbf{T}_S = -\gamma(\nabla \cdot \mathbf{n})\mathbf{n}|_{r=R+\xi(\theta,\phi)}$$

$$\begin{aligned}\mathbf{n} &= n_r \mathbf{i}_r + n_\theta \mathbf{i}_\theta + n_\phi \mathbf{i}_\phi = \mathbf{i}_r - \frac{1}{R} \frac{\partial \xi}{\partial \theta} \mathbf{i}_\theta - \frac{1}{R \sin \theta} \frac{\partial \xi}{\partial \phi} \mathbf{i}_\phi \\ n_r &= 1, \quad n_\theta = -\frac{1}{R} \frac{\partial \xi}{\partial \theta}, \quad n_\phi = -\frac{1}{R \sin \theta} \frac{\partial \xi}{\partial \phi}\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{n} &= \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta n_\theta) + \frac{1}{r \sin \theta} \frac{\partial n_\phi}{\partial \phi} \right]_{r=R+\xi} \\ &= \frac{2}{R} - \frac{2\xi}{R^2} - \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \xi}{\partial \theta} \right) - \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 \xi}{\partial \phi^2}\end{aligned}$$

$$\xi(\theta, \phi) = \Re[\hat{\xi} P_n^m(\cos \theta) e^{-jm\phi}]$$

$$\begin{aligned}T_{sr} &= -\gamma \nabla \cdot \mathbf{n} n_r \\ &= -\gamma \nabla \cdot \mathbf{n} \\ &= -\gamma \left[\frac{2}{R} - \frac{2\xi}{R^2} - \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \xi}{\partial \theta} \right) - \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 \xi}{\partial \phi^2} \right] \\ \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP_n^m(\cos \theta)}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} P_n^m(\cos \theta) &= -n(n+1)P_n^m(\cos \theta) \\ T_{sr} &= -\frac{2\gamma}{R} + \Re \left[\frac{\gamma \hat{\xi}}{R^2} \left[2P_n^m(\cos \theta) + \underbrace{\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP_n^m(\cos \theta)}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} P_n^m(\cos \theta)}_{-n(n+1)P_n^m(\cos \theta)} \right] e^{-jm\phi} \right]\end{aligned}$$

$$\begin{aligned} T_{sr} &= -\frac{2\gamma}{R} + \Re \left[\frac{\gamma \hat{\xi}}{R^2} P_n^m(\cos \theta) (2 - n(n+1)) \right] \\ &= -\frac{2\gamma}{R} - \Re \left[\frac{\gamma \hat{\xi}}{R^2} P_n^m(\cos \theta) (n-1)(n+2) \right] \end{aligned}$$

$$\begin{aligned} T_{s\theta} &= -\gamma \nabla \cdot \mathbf{n} n_\theta \\ &= -\gamma \underbrace{\nabla \cdot \mathbf{n}}_{\text{zero-order}, 2/R} \underbrace{\left(-\frac{1}{R} \frac{\partial \xi}{\partial \theta} \right)}_{\text{first-order}} \\ &= \frac{2\gamma}{R^2} \Re \left[\hat{\xi} \frac{dP_n^m(\cos \theta)}{d\theta} e^{-jm\phi} \right] \end{aligned}$$

$$\begin{aligned} T_{s\phi} &= -\gamma \nabla \cdot \mathbf{n} n_\phi \\ &= -\gamma \underbrace{\nabla \cdot \mathbf{n}}_{\text{zero order}, 2/R} \underbrace{\left(-\frac{1}{R \sin \theta} \frac{\partial \xi}{\partial \phi} \right)}_{\text{first order}} \\ &= \frac{2\gamma m}{R^2 \sin \theta} \Re[-j\hat{\xi} P_n^m(\cos \theta) e^{-jm\phi}] \end{aligned}$$

Identities: $P_n^{m=0}(x) = \frac{1}{2^n n!} \frac{d^n(x^2 - 1)^n}{dx^n}$

$$P_n^m(x) = (x^2 - 1)^{m/2} \frac{d^m P_n^{m=0}(x)}{dx^m}$$