6.642 Continuum Electromechanics
Fall 2008

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Linearized Surface Tension Force Density in Spherical Coordinates

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$$\mathbf{T_S} = -\gamma(\nabla\cdot\mathbf{n})\mathbf{n}\big|_{r=R+\xi(\theta,\phi)}$$

$$\mathbf{n} = n_r\mathbf{i_r} + n_\theta\mathbf{i}_\theta + n_\phi\mathbf{i}_\phi = \mathbf{i_r} - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\mathbf{i}_\theta - \frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\mathbf{i}_\phi$$

$$n_r = 1, \quad n_\theta = -\frac{1}{R}\frac{\partial\xi}{\partial\theta}, \quad n_\phi = -\frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}$$

$$\begin{aligned}\nabla\cdot\mathbf{n} &= \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 n_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta n_\theta) + \frac{1}{r\sin\theta}\frac{\partial n_\phi}{\partial\phi}\right]_{r=R+\xi}\\ &= \frac{2}{R} - \frac{2\xi}{R^2} - \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right) - \frac{1}{R^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}\end{aligned}$$

$$\xi(\theta,\phi) = \Re[\hat{\xi}P_n^m(\cos\theta)e^{-jm\phi}]$$

$$\begin{aligned}T_{sr} &= -\gamma\nabla\cdot\mathbf{n}n_r\\ &= -\gamma\nabla\cdot\mathbf{n}\\ &= -\gamma\left[\frac{2}{R} - \frac{2\xi}{R^2} - \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right) - \frac{1}{R^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}\right]\end{aligned}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n^m(\cos\theta)}{d\theta}\right) - \frac{m^2}{\sin^2\theta}P_n^m(\cos\theta) = -n(n+1)P_n^m(\cos\theta)$$

$$T_{sr} = -\frac{2\gamma}{R} + \Re\left[\frac{\gamma\hat{\xi}}{R^2}\left[2P_n^m(\cos\theta) + \underbrace{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n^m(\cos\theta)}{d\theta}\right) - \frac{m^2}{\sin^2\theta}P_n^m(\cos\theta)}_{-n(n+1)P_n^m\cos\theta}\right]e^{-jm\phi}\right]$$

$$
\begin{aligned}
T_{sr} &= -\frac{2\gamma}{R} + \Re\left[\frac{\gamma\hat{\xi}}{R^2} P_n^m(\cos\theta)(2 - n(n+1))\right] \\
&= -\frac{2\gamma}{R} - \Re\left[\frac{\gamma\hat{\xi}}{R^2} P_n^m(\cos\theta)(n-1)(n+2)\right] \\
T_{s\theta} &= -\gamma \nabla\cdot\mathbf{n} n_\theta \\
&= -\gamma \underbrace{\nabla\cdot\mathbf{n}}_{\text{zero-order},2/R} \underbrace{\left(-\frac{1}{R}\frac{\partial\xi}{\partial\theta}\right)}_{\text{first-order}} \\
&= \frac{2\gamma}{R^2}\Re\left[\hat{\xi}\frac{dP_n^m(\cos\theta)}{d\theta}e^{-jm\phi}\right] \\
T_{s\phi} &= -\gamma\nabla\cdot\mathbf{n} n_\phi \\
&= -\gamma \underbrace{\nabla\cdot\mathbf{n}}_{\text{zero order},2/R} \underbrace{\left(-\frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\right)}_{\text{first order}} \\
&= \frac{2\gamma m}{R^2\sin\theta}\Re[-j\hat{\xi}P_n^m(\cos\theta)e^{-jm\phi}]
\end{aligned}
$$

Identities: $P_n^{m=0}(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}$

$$P_n^m(x) = (x^2-1)^{m/2}\frac{d^m P_n^{m=0}(x)}{dx^m}$$