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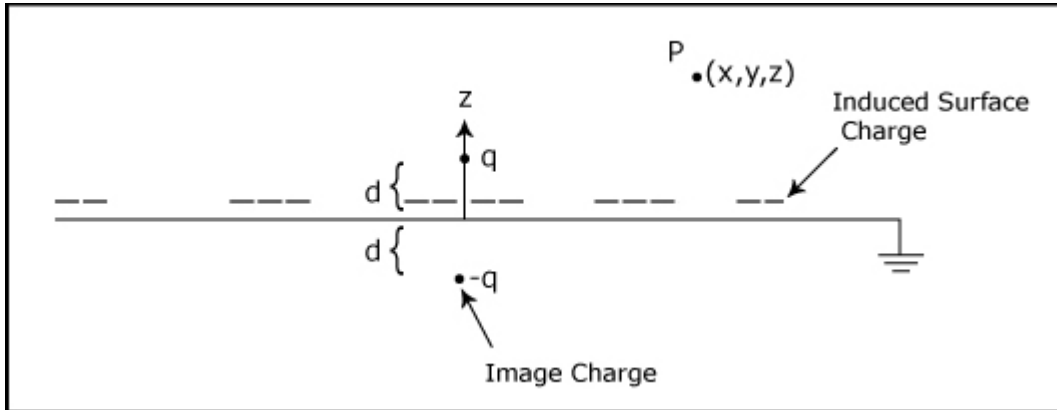
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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6.641, Electromagnetic Fields, Forces, and Motion
 Prof. Markus Zahn
Lecture 5: Method of Images

I. Point Charge Above Ground Plane

1. Potential and Electric Field



$$\Phi_p = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right]$$

$$\vec{E}_p = -\nabla\Phi_p = - \left[\frac{\partial\Phi_p}{\partial x} \vec{i}_x + \frac{\partial\Phi_p}{\partial y} \vec{i}_y + \frac{\partial\Phi_p}{\partial z} \vec{i}_z \right]$$

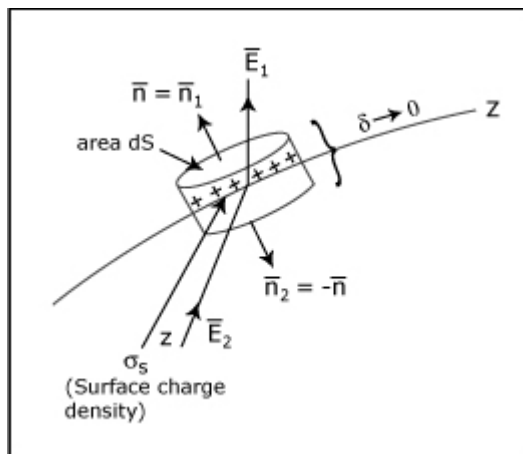
$$= \frac{q}{4\pi\epsilon_0} \left[\frac{\cancel{z} \left(x \vec{i}_x + y \vec{i}_y + (z-d) \vec{i}_z \right)}{\cancel{z} \left[x^2 + y^2 + (z-d)^2 \right]^{3/2}} - \frac{\cancel{z} \left(x \vec{i}_x + y \vec{i}_y + (z+d) \vec{i}_z \right)}{\cancel{z} \left[x^2 + y^2 + (z+d)^2 \right]^{3/2}} \right]$$

$$\vec{E}_p(z=0) = \frac{q}{2\pi\epsilon_0} \frac{(-d)}{\left[x^2 + y^2 + d^2 \right]^{3/2}} \vec{i}_z$$

(perpendicular to equipotential ground plane)

2. Gauss's Law Boundary Condition

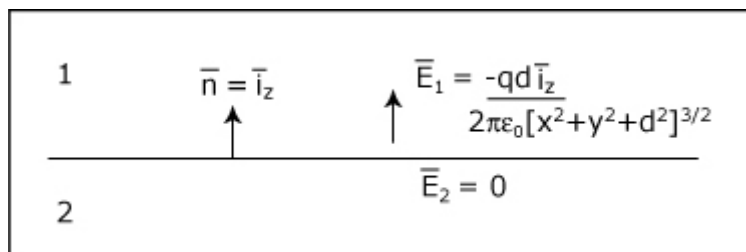
$$\oint_S \epsilon_0 \vec{E} \cdot \vec{da} = \int_V \rho dV$$



$$\oint_S \epsilon_0 \vec{E} \cdot \vec{da} = (\epsilon_0 \vec{E}_1 \cdot \vec{n}_1 + \epsilon_0 \vec{E}_2 \cdot \vec{n}_2) dS = \sigma_s dS \quad (\text{total charge inside pillbox})$$

$$\sigma_s = \epsilon_0 \vec{n} \cdot [\vec{E}_1 - \vec{E}_2]$$

3. Back to Point Charge Above Ground Plane



At $z=0$:

$$\sigma_s = \epsilon_0 \vec{n} \cdot [\vec{E}_1 - \vec{E}_2] = \epsilon_0 \vec{i}_z \cdot \vec{E}_1 = \epsilon_0 E_z = \frac{-qd}{2\pi [x^2 + y^2 + d^2]^{3/2}} = \frac{-qd}{2\pi [r^2 + d^2]^{3/2}}$$

$$r^2 = x^2 + y^2$$

$$q_T(z=0) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} \sigma_s dx dy = \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \sigma_s r dr d\phi = \frac{-qd}{2\pi} \int_{r=0}^{\infty} \frac{r dr}{[r^2 + d^2]^{3/2}}$$

$$u = r^2 + d^2 \Rightarrow du = 2rdr$$

$$\int \frac{rdr}{[r^2 + d^2]^{3/2}} = \int \frac{du}{2u^{3/2}} = -u^{-1/2} = -\frac{1}{\sqrt{r^2 + d^2}}$$

$$q_T(z=0) = \frac{+qd}{\sqrt{r^2 + d^2}} \Big|_0^\infty = -q$$

$$\bar{f}_q = \frac{-q^2}{4\pi\epsilon_0(2d)^2} \bar{i}_z = \frac{-q^2}{16\pi\epsilon_0 d^2} \bar{i}_z$$

II. Point Charge and Sphere

1. Grounded Sphere

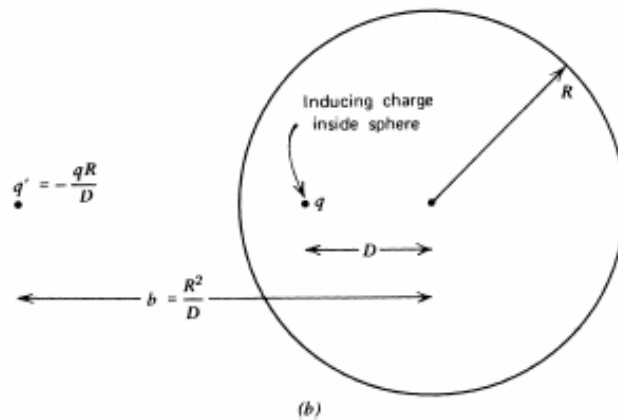
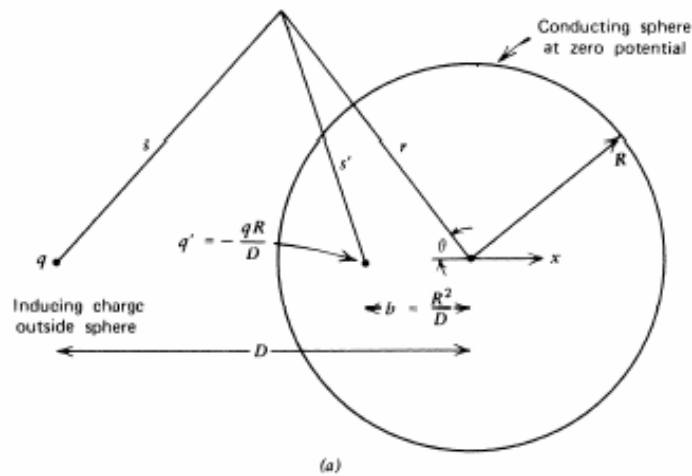


Figure 2-27 (a) The field due to a point charge q , a distance D outside a conducting sphere of radius R , can be found by placing a single image charge $-qR/D$ at a distance $b = R^2/D$ from the center of the sphere. (b) The same relations hold true if the charge q is inside the sphere but now the image charge is outside the sphere, since $D < R$.

$$\Phi = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{s} + \frac{q'}{s'} \right)$$

$$s = [r^2 + D^2 - 2rD \cos \theta]^{1/2}, \quad s' = [b^2 + r^2 - 2rb \cos \theta]^{1/2}$$

$$\Phi(r=R) = 0 \Rightarrow \frac{q}{s} = -\frac{q'}{s'} \Rightarrow \left(\frac{q}{s} \right)^2 = \left(\frac{q'}{s'} \right)^2$$

$$q^2 s'^2 = q'^2 s^2 \Rightarrow q'^2 [R^2 + D^2 - 2RD \cos \theta] = q^2 [b^2 + R^2 - 2Rb \cos \theta]$$

$$q'^2 (R^2 + D^2) = q^2 (b^2 + R^2)$$

$$+q'^2 \cancel{2RD \cos \theta} = +q^2 \cancel{2Rb \cos \theta} \Rightarrow \frac{q'^2}{q^2} = \frac{b}{D}$$

$$\frac{b}{D} (R^2 + D^2) = b^2 + R^2 \Rightarrow b^2 - b \left(\frac{R^2}{D} + D \right) + R^2 = 0$$

$$(b-D) \left(b - \frac{R^2}{D} \right) = 0$$

$$b = \frac{R^2}{D}$$

$$q'^2 = q^2 \frac{b}{D} = q^2 \frac{R^2}{D^2} \Rightarrow q' = -qR/D$$

force on sphere

$$f_x = \frac{qq'}{4\pi\epsilon_0 (D-b)^2} = \frac{-q^2 R/D}{4\pi\epsilon_0 \left(D - \frac{R^2}{D} \right)^2} = \frac{-q^2 R D}{4\pi\epsilon_0 (D^2 - R^2)^2}$$

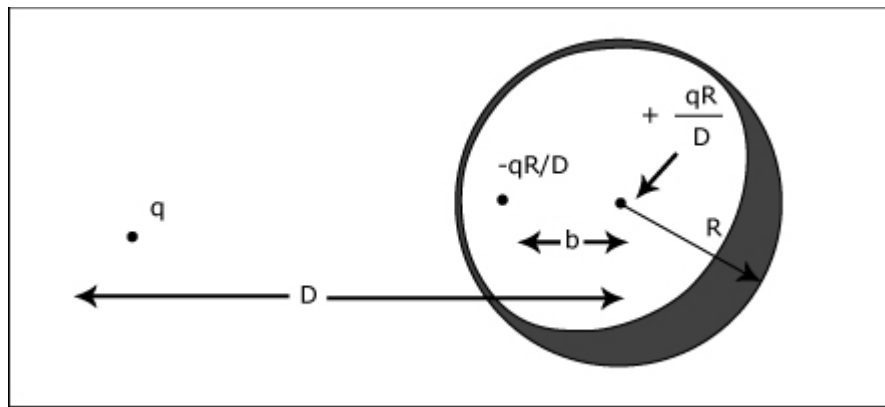
2. Isolated Sphere [Put additional Image Charge $+q' = +qR/D$ at center]
(zero charge)

$$\Phi(r=R) = \frac{q'}{4\pi\epsilon_0 R} = \frac{q}{4\pi\epsilon_0 D}$$

force on sphere

$$f_x = \frac{q}{4\pi\epsilon_0} \left[\frac{q'}{(D-b)^2} - \frac{q'}{D^2} \right] = \frac{qq' [D^2 - (D-b)^2]}{4\pi\epsilon_0 D^2 (D-b)^2} = \frac{-q^2 R [2bD - b^2]}{4\pi\epsilon_0 D^3 \left(D - \frac{R^2}{D} \right)^2}$$

$$f_x = \frac{-q^2 R D^2}{4\pi\epsilon_0 D^3 (D^2 - R^2)^2} \frac{R^2}{D} \left[2D - \frac{R^2}{D} \right] = \frac{-q^2 R^3}{4\pi\epsilon_0 D^3 (D^2 - R^2)^2} [2D^2 - R^2]$$



III. Demonstration 4.7.1 – Charge Induced in Ground Plane by Overhead Conductor

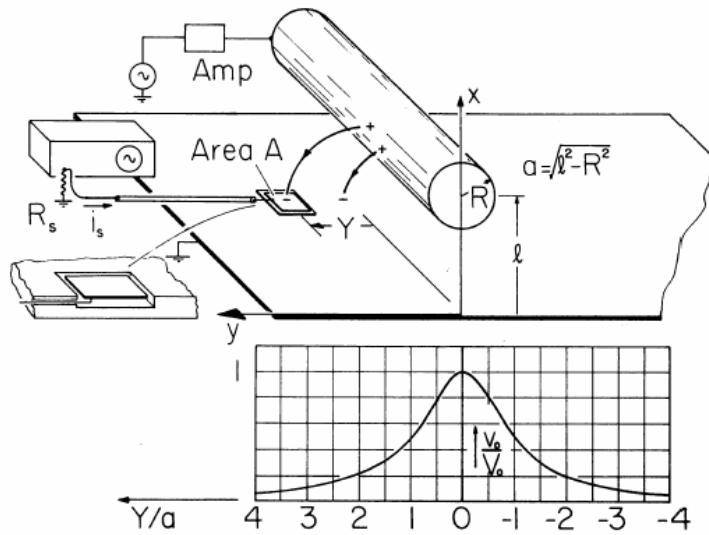
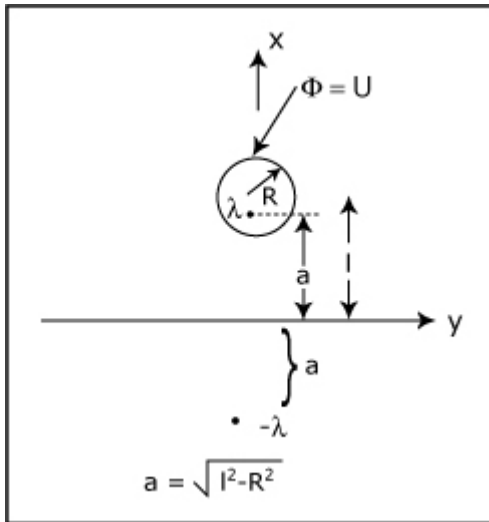


Figure 4.7.2 Charge induced on ground plane by overhead conductor is measured by probe. Distribution shown is predicted by (4.7.7).

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$$\Phi = \frac{-\lambda}{2\pi\epsilon_0} \ln \left[\frac{[(a-x)^2 + y^2]^{1/2}}{[(a+x)^2 + y^2]^{1/2}} \right] = \frac{-\lambda}{4\pi\epsilon_0} \ln \left[\frac{(a-x)^2 + y^2}{(a+x)^2 + y^2} \right]$$

$$C' = \frac{\lambda}{\Phi(x=l-R, y=0)} = \frac{\lambda}{\frac{-\lambda}{2\pi\epsilon_0} \ln \frac{a-l+R}{a+l-R}} = \frac{2\pi\epsilon_0}{\ln \left[\frac{\sqrt{l^2 - R^2} + l}{R} \right]}, \quad \Phi(x=l-R, y=0) = U$$

$$\begin{aligned}
\sigma_s = \epsilon_0 E_x (x = 0) &= -\epsilon_0 \left. \frac{\partial \Phi}{\partial x} \right|_{x=0} \\
&= \frac{+\cancel{\epsilon_0} \lambda}{4\pi \cancel{\epsilon_0}} \frac{d}{dx} \left[\ln \left[(a-x)^2 + y^2 \right] - \ln \left[(a+x)^2 + y^2 \right] \right] \\
&= \frac{\lambda}{4\pi} \left[\frac{-2(a-x)}{(a-x)^2 + y^2} - \frac{2(a+x)}{(a+x)^2 + y^2} \right] \Bigg|_{x=0} \\
&= \frac{-\lambda a}{\pi(a^2 + y^2)}
\end{aligned}$$

Total Charge per unit length on ground plane is:

$$\begin{aligned}
\lambda_T (x = 0) &= \int_{y=-\infty}^{\infty} \sigma_s dy = \int_{-\infty}^{\infty} \frac{-\lambda a}{\pi(a^2 + y^2)} dy = \frac{-\lambda \cancel{a}}{\pi} \frac{1}{\cancel{a}} \underbrace{\tan^{-1} \frac{y}{a}}_{\pi} \Bigg|_{-\infty}^{\infty} \\
&= -\lambda
\end{aligned}$$

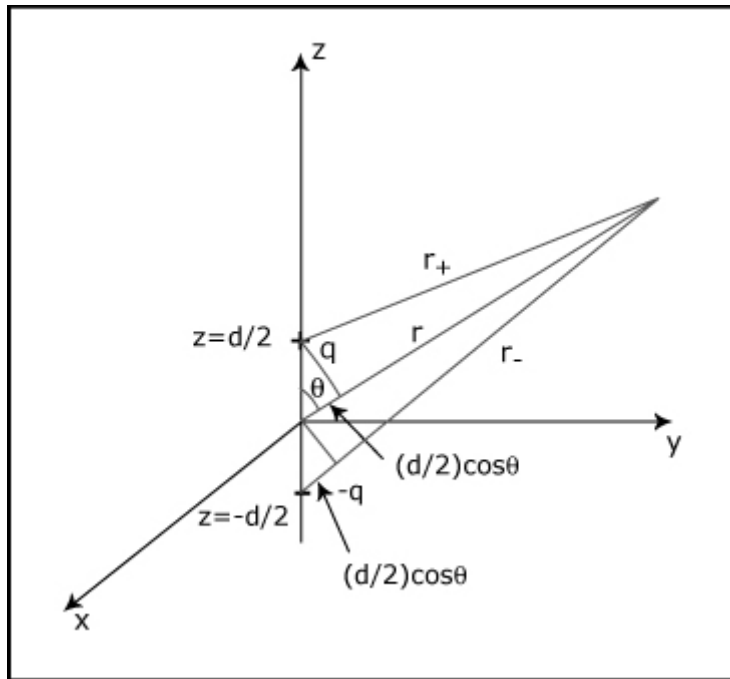
$$i_s = \frac{dq}{dt} \approx A \frac{d\sigma_s}{dt} = \frac{-aA}{\pi(a^2 + y^2)} \frac{d\lambda}{dt} = \frac{-aAC'}{\pi(a^2 + y^2)} \frac{dU}{dt}$$

take $U = U_0 \cos \omega t$

$$v_0 = -i_s R_s = -\frac{C' A a}{\pi(a^2 + y^2)} U_0 \omega \sin \omega t$$

IV. Point Electric Dipole

1. Potential



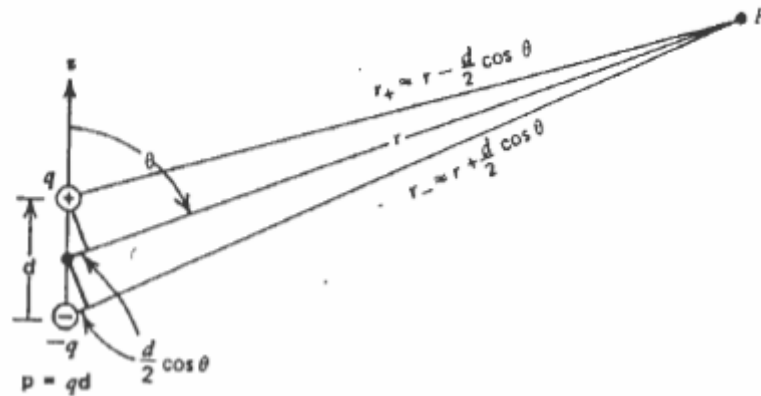
$$\Phi = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r_+} - \frac{1}{r_-} \right]$$

$$r_+ = \sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}$$

$$r_- = \sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}$$

Note: $\Phi(z = 0) = 0$

2. Point Electric Dipole ($r \gg d$)



$$r_+ \approx r - \frac{d}{2} \cos \theta \approx r \left[1 - \frac{d}{2r} \cos \theta \right]$$

$$r_- \approx r + \frac{d}{2} \cos \theta \approx r \left[1 + \frac{d}{2r} \cos \theta \right]$$

$$\Phi \approx \frac{q}{4\pi\epsilon_0 r} \left[\frac{1}{1 - \frac{d}{2r} \cos \theta} - \frac{1}{1 + \frac{d}{2r} \cos \theta} \right] \approx \frac{q}{4\pi\epsilon_0 r} \left[1 + \frac{d}{2r} \cos \theta - \left(1 - \frac{d}{2r} \cos \theta \right) \right]$$

$$\approx \frac{qd \cos \theta}{4\pi\epsilon_0 r^2}$$

$$\lim_{\substack{d \rightarrow 0 \\ q \rightarrow \infty}} p = qd \text{ (dipole moment)} \Rightarrow \Phi \approx \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = -\nabla\Phi = - \left[\frac{\partial\Phi}{\partial r} \vec{i}_r + \frac{1}{r} \frac{\partial\Phi}{\partial\theta} \vec{i}_\theta + \frac{1}{r \sin\theta} \frac{\partial\Phi}{\partial\phi} \vec{i}_\phi \right]$$

$$= \frac{p}{4\pi\epsilon_0 r^3} \left[2 \cos \theta \vec{i}_r + \sin \theta \vec{i}_\theta \right]$$

3. Field Lines: $\frac{dr}{rd\theta} = \frac{E_r}{E_\theta} = \frac{2 \cos \theta}{\sin \theta} = 2 \cot \theta$

$$\frac{dr}{r} = 2 \cot \theta d\theta \Rightarrow \ln r = 2 \ln(\sin \theta) + C$$

$$r = r_0 \sin^2 \theta$$

$$r_0 = r \left(\theta = \frac{\pi}{2} \right)$$

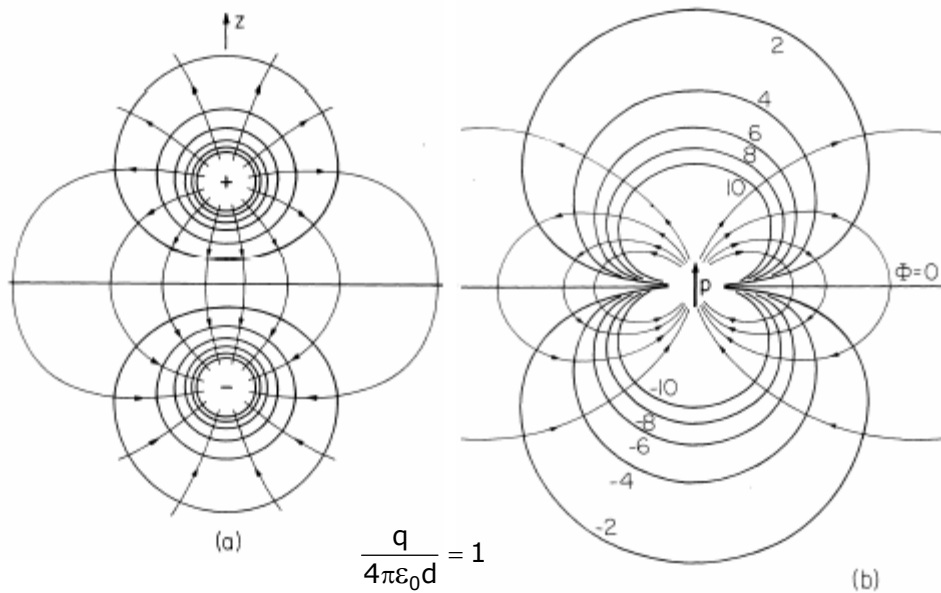


Figure 4.4.2 (a) Cross section of equipotentials and lines of electric field intensity for the two charges of Figure 4.4.1. (b) Limit in which pair of charges form a dipole at the origin.

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V. Line Current Above a Perfect Conductor

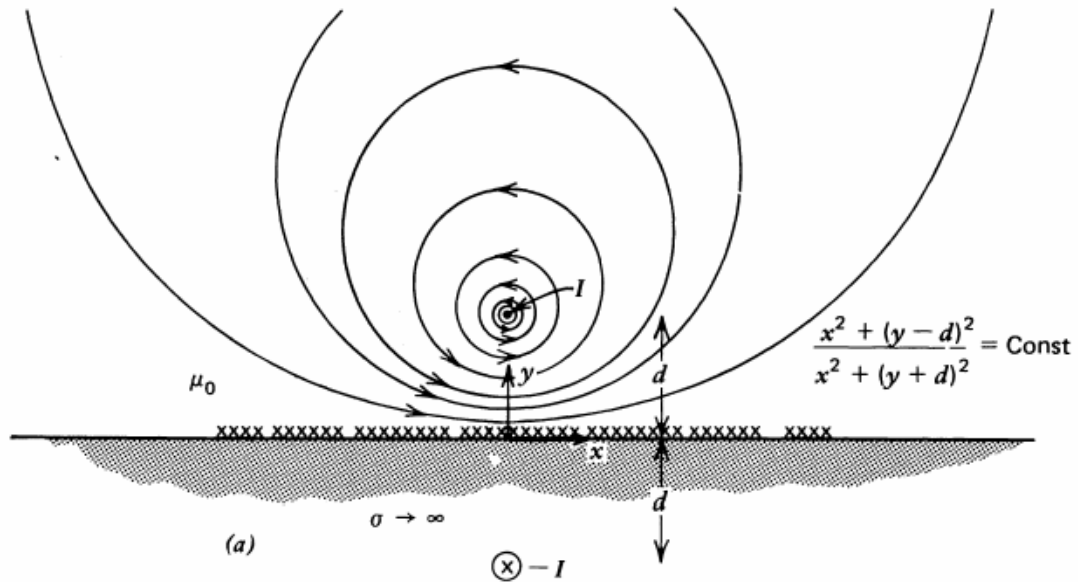


Figure 5-24 (a) A line current above a perfect conductor induces an oppositely directed surface current that is equivalent to a symmetrically located image line current.

$$\bar{f}_I = \bar{I} \times (\mu_0 \bar{H}) \quad \text{Newton/meter [force per unit length]}$$

$$= I \bar{i}_z \times \left(\mu_0 \frac{I}{4\pi d} \bar{i}_x \right) = \frac{\mu_0 I^2}{4\pi d} \bar{i}_y$$