Lecture 4: The Scalar Electric Potential and the Coulomb Superposition Integral

I. <u>Quasistatics</u>

Electroquasistatics (EQS)   Magnetquasistatics (MQS)

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t}\left(\mu_0 \bar{H}\right)^{\nearrow 0} \approx 0$$

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t}\left(\mu_0 \bar{H}\right)$$

$$\nabla \cdot \left(\varepsilon_0 \bar{E}\right) = \rho$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial}{\partial t}\left(\varepsilon_0 \bar{E}\right)^{\nearrow 0}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial}{\partial t}\left(\varepsilon_0 \bar{E}\right)$$

$$\nabla \cdot \left(\mu_0 \bar{H}\right) = 0$$

$$\nabla \cdot \bar{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot \bar{J} = 0$$

$$\nabla \cdot \left(\varepsilon_0 \bar{E}\right) = \rho$$

II. <u>Irrotational EQS Electric Field</u>

    1. Conservative Electric Field

$$\oint_C \bar{E} \cdot \overline{ds} = -\frac{d}{dt}\int_S \mu_0 \bar{H} \cdot \overline{da}^{\nearrow 0} \approx 0$$
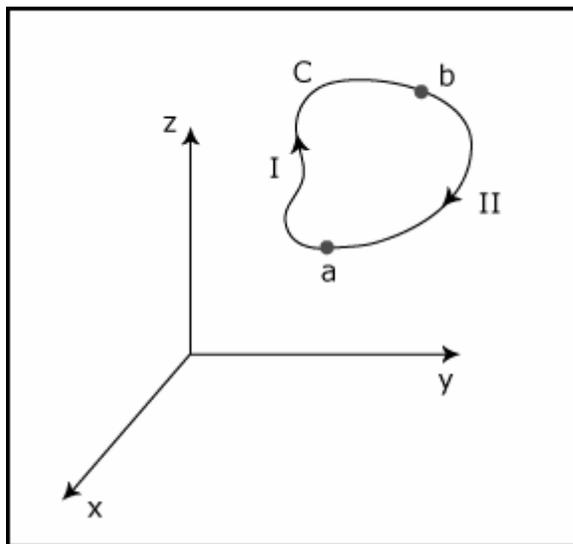
$$\oint_C \overline{E} \cdot \overline{ds} = \int_a^b \overline{E} \cdot \overline{ds} + \int_b^a \overline{E} \cdot \overline{ds} = 0 \Rightarrow \underbrace{\int_a^b \overline{E} \cdot \overline{ds}}_{\substack{I \\ \text{Electromotive Force} \\ \text{(EMF)}}} = \int_a^b \overline{E} \cdot \overline{ds}$$

$$\begin{matrix} I & II & & I & II \end{matrix}$$

EMF between 2 points (a, b) independent of path
$\overline{E}$ field is <u>conservative</u>

$$\underset{\nearrow \text{ Scalar}}{\Phi\left(\overline{r}\right)} - \Phi\left(\overline{r}_{ref}\right) = \int_r^{\overline{r}_{ref}} \overline{E} \cdot \overline{ds}$$

electric potential

$$\int_a^b \overline{E} \cdot \overline{ds} = \int_a^{\overline{r}_{ref}} \overline{E} \cdot \overline{ds} + \int_{\overline{r}_{ref}}^b \overline{E} \cdot \overline{ds} = \Phi(a) - \Phi\left(\overline{r}_{ref}\right) + \Phi\left(\overline{r}_{ref}\right) - \Phi(b) = \Phi(a) - \Phi(b)$$

2. The Electric Scalar Potential
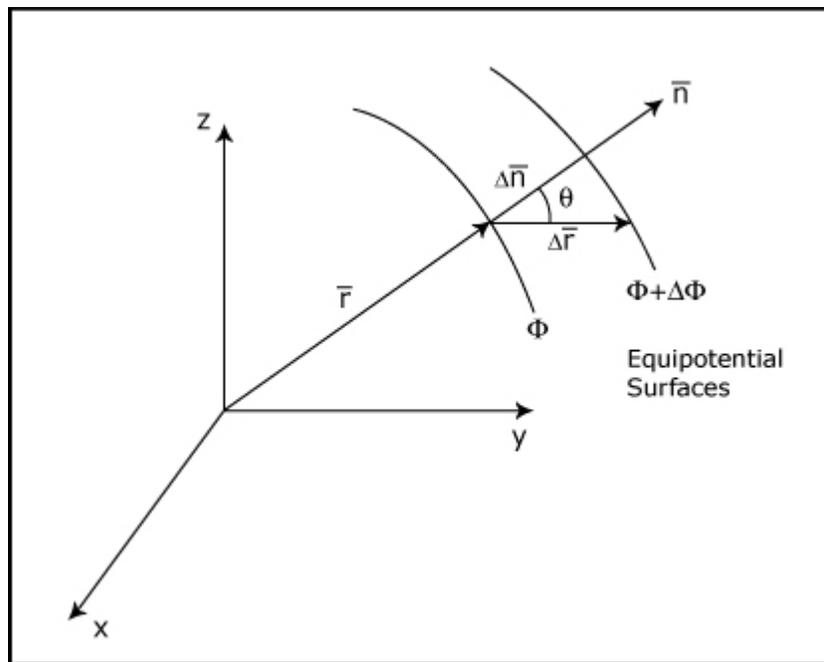
$$\overline{r} = x\,\overline{i}_x + y\,\overline{i}_y + z\,\overline{i}_z$$

$$\Delta\overline{r} = \Delta x\,\overline{i}_x + \Delta y\,\overline{i}_y + \Delta z\,\overline{i}_z$$

$$\Delta n = \Delta r \cos\theta$$

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$$\Delta\Phi = \Phi\left(x + \Delta x, y + \Delta y, z + \Delta z\right) - \Phi\left(x, y, z\right)$$

$$= \Phi\left(x, y, z\right) + \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z - \Phi\left(x, y, z\right)$$

$$= \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z$$

$$= \underbrace{\left[\frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y + \frac{\partial\Phi}{\partial z}\bar{i}_z\right]}_{\text{grad } \Phi = \nabla\Phi} \bullet \Delta\bar{r}$$

$$\nabla = \bar{i}_x\frac{\partial}{\partial x} + \bar{i}_y\frac{\partial}{\partial y} + \bar{i}_z\frac{\partial}{\partial z}$$

$$\text{grad } \Phi = \nabla\Phi = \bar{i}_x\frac{\partial\Phi}{\partial x} + \bar{i}_y\frac{\partial\Phi}{\partial y} + \bar{i}_z\frac{\partial\Phi}{\partial z}$$

$$\int_{\bar{r}}^{\bar{r}+\Delta\bar{r}} \bar{E}\bullet\overline{ds} = \Phi\left(\bar{r}\right) - \Phi\left(\bar{r} + \Delta\bar{r}\right) = -\Delta\Phi = -\nabla\Phi\bullet\Delta\bar{r} = \bar{E}\bullet\Delta\bar{r}$$

$$\boxed{\bar{E} = -\nabla\Phi}$$

$$\Delta\Phi = \frac{\Delta\Phi}{\Delta n}\Delta r\cos\theta = \frac{\Delta\Phi}{\Delta n}\bar{n}\bullet\Delta\bar{r} = \nabla\Phi\bullet\Delta\bar{r}$$

$$\nabla\Phi = \frac{\Delta\Phi}{\Delta n}\bar{n} = \frac{\partial\Phi}{\partial n}\bar{n}$$

The gradient is in the direction perpendicular to the equipotential surfaces.

III. Vector Identity

$$\nabla \times \bar{E} = 0$$

$$\bar{E} = -\nabla\Phi$$

$$\nabla \times \left(\nabla\Phi\right) = 0$$

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IV. Sample Problem

$$\Phi(x, y) = \frac{V_0 \, xy}{a^2} \qquad \text{(Equipotential lines hyperbolas: xy=constant)}$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial x} \bar{i}_x + \frac{\partial\Phi}{\partial y} \bar{i}_y \right]$$

$$= \frac{-V_0}{a^2} \left(y\,\bar{i}_x + x\,\bar{i}_y \right)$$

Electric Field Lines [lines tangent to electric field]

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y} \Rightarrow ydy = xdx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 - x^2 = y_0^2 - x_0^2 \qquad \text{[lines pass through point } (x_0, y_0)]$$
(hyperbolas orthogonal to xy)
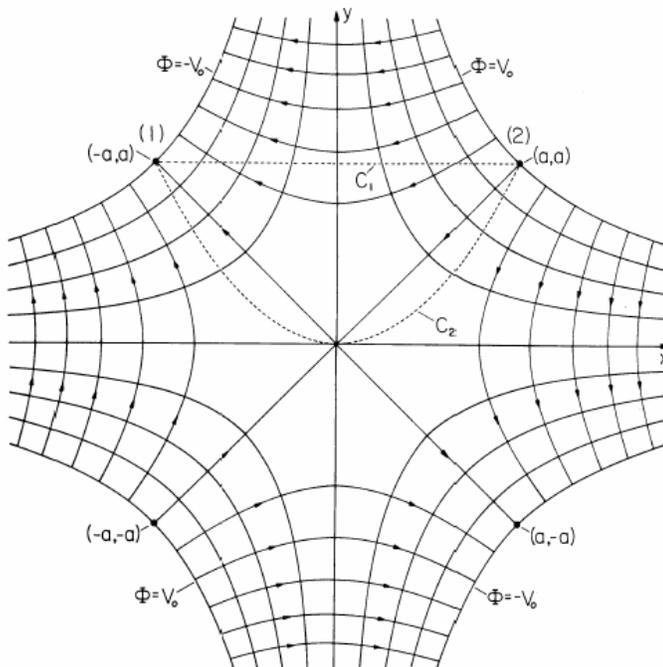
Figure 4.1.3 Cross-sectional view
of surfaces of constant potential for
two-dimensional potential given by (18).

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

V. Poisson's Equation

$$\nabla \cdot \bar{E} = \nabla \cdot (-\nabla \Phi) = \rho / \varepsilon_0 \Rightarrow \nabla^2 \Phi = -\rho / \varepsilon_0$$

$$\nabla^2 \Phi = \nabla \cdot (\nabla \Phi) = \left[\bar{i}_x \frac{\partial}{\partial x} + \bar{i}_y \frac{\partial}{\partial y} + \bar{i}_z \frac{\partial}{\partial z} \right] \cdot \left[\frac{\partial \Phi}{\partial x} \bar{i}_x + \frac{\partial \Phi}{\partial y} \bar{i}_y + \frac{\partial \Phi}{\partial z} \bar{i}_z \right]$$

$$= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

## VI. Coulomb Superposition Integral

1. Point Charge

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{q}{4\pi \varepsilon_0 r^2} \Rightarrow \Phi = \frac{q}{4\pi \varepsilon_0 r} + C$$

Take reference $\Phi(r \to \infty) = 0 \Rightarrow C = 0$

$$\Phi = \frac{q}{4\pi \varepsilon_0 r}$$

2. Superposition of Charges

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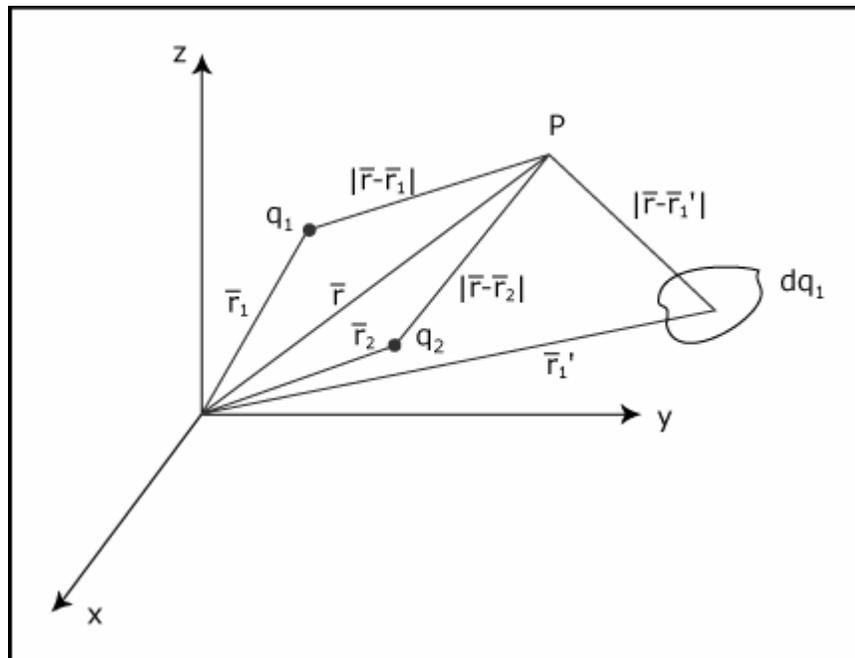
$$d\Phi_T(P) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{\left|\bar{r}-\bar{r}_1\right|} + \frac{q_2}{\left|\bar{r}-\bar{r}_2\right|} + \ldots \frac{dq_1}{\left|\bar{r}-\bar{r}_1'\right|} + \frac{dq_2}{\left|\bar{r}-\bar{r}_2'\right|} + \ldots\right]$$

$$\Phi_T(P) = \frac{1}{4\pi\varepsilon_0}\left[\sum_{n=1}^{N}\frac{q_n}{\left|\bar{r}-\bar{r}_n\right|} + \int_{\substack{\text{all line,}\\\text{surface, and}\\\text{volume charges}}}\frac{dq}{\left|\bar{r}-\bar{r}'\right|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\sum_{n=1}^{N}\frac{q_n}{\left|\bar{r}-\bar{r}_n\right|} + \int_L\frac{\lambda(\bar{r}')dl'}{\left|\bar{r}-\bar{r}'\right|} + \int_S\frac{\sigma_s(\bar{r}')da'}{\left|\bar{r}-\bar{r}'\right|} + \int_V\frac{\rho(\bar{r}')dV'}{\left|\bar{r}-\bar{r}'\right|}\right]$$

Short-hand notation

$$\Phi(\bar{r}) = \int_V\frac{\rho(\bar{r}')dV'}{4\pi\varepsilon_0\left|\bar{r}-\bar{r}'\right|}$$

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