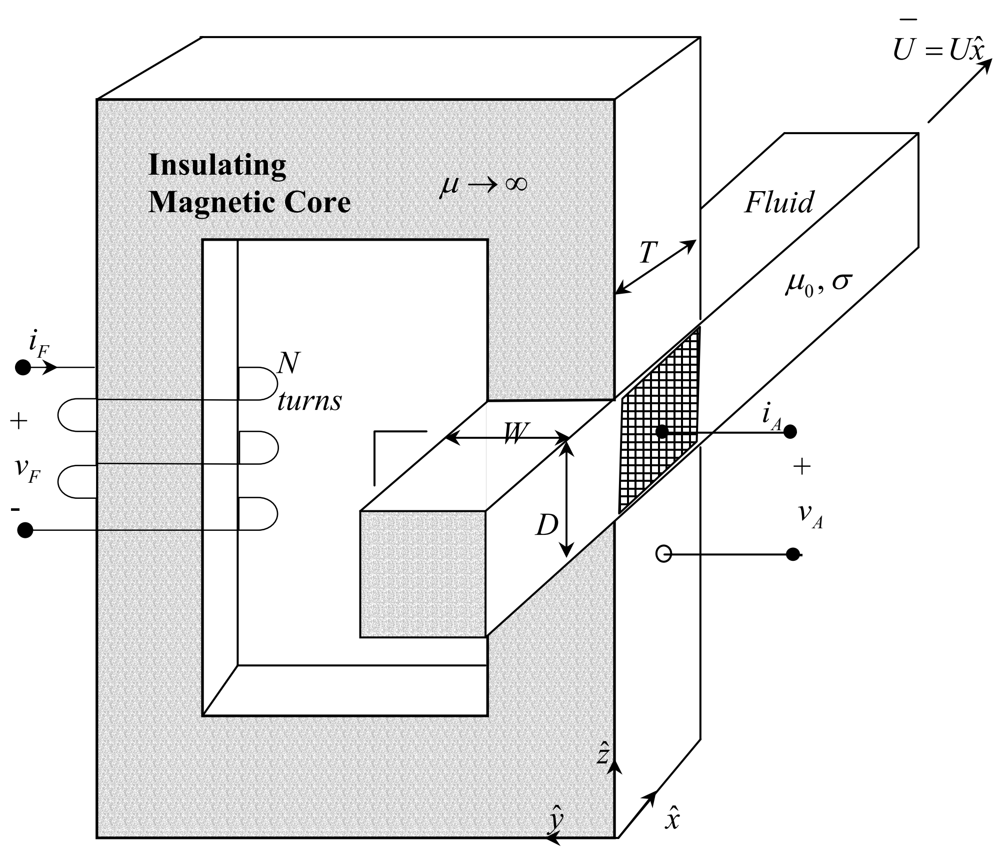
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion

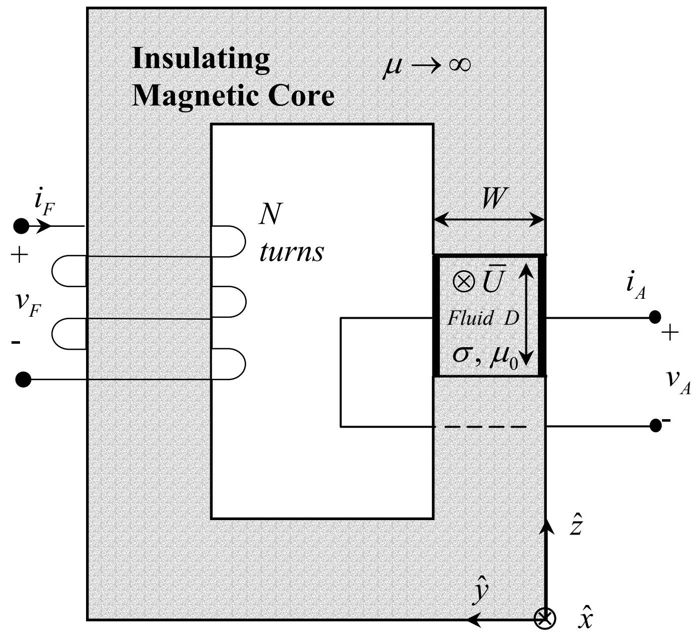
Final Exam May 20, 2003
Spring Term 2003, 1:30-4:30PM

6.641 Formula Sheet Attached in the study materials section. You are also allowed to use the two formula sheets that you prepared for Quiz 1 and 2 plus an additional 8 ½"x 11" formula sheet (both sides) that you have prepared for the Final Exam.

Problem 1 (20 points)

The two figures shown below present two different views of a magnetohydrodynamic generator. In this generator, a fluid having conductivity σ and free-space permeability is pumped through a rectangular channel with velocity U in the $\hat{x}$ direction. The width and height of the channel are W and D, respectively. The channel passes through the gap of a perfectly-permeable C-core of width T in the $\hat{x}$ direction. The C-core is excited by a perfectly-conducting N-turn field coil that carries the current i_F and has a terminal voltage v_F. The two side walls of the channel make perfect electrical contact with the fluid over the width T as the channel passes through the C-core. The current through these armature contacts is i_A and the voltage across them is v_A.

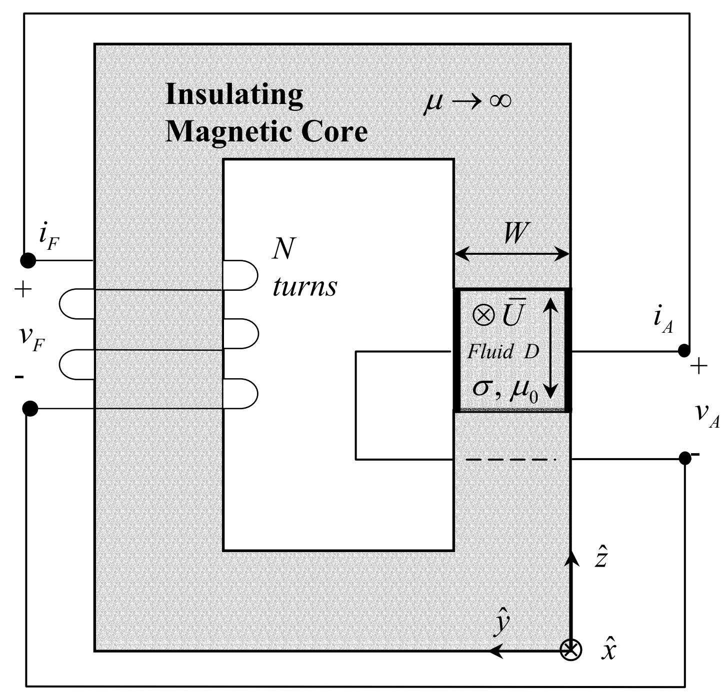




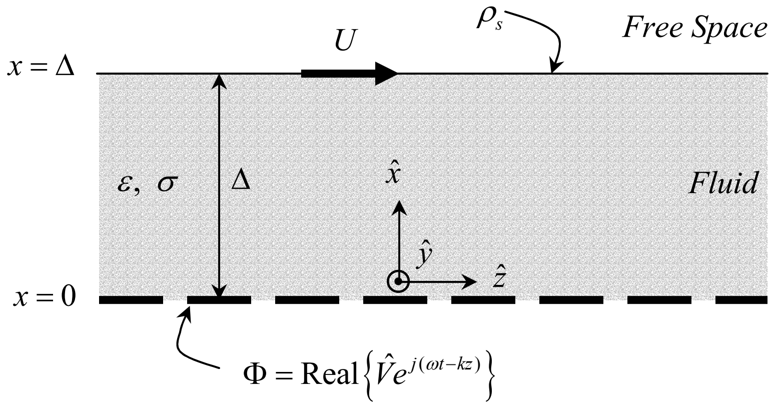
a) Determine the $\hat{z}$ -directed magnetic flux density in the gap of the C-core in terms of the field current i_F , and the parameters of the generator. Make reasonable magnetic circuit approximations, and ignore the flux density sourced by the armature current.
b) Determine the self-inductance of the field coil in terms of the parameters of the generator.
c) The static terminal relation for the armature takes the form

$$v_A = Ri_A + GUi_F$$

Determine R and G in terms of the parameters of the generator.
d) Determine the mechanical power that is required to pump the fluid through the channel in terms of i_A, i_F, U and the parameters of the generator.
e) The generator is connected such that $i_F = -i_A$ and $v_F = v_A$, as shown below, in an effort to produce self-excitation. For what range of U will it exhibit such self-excitation? Ignore any armature inductance.



Problem 2 (20 points)



A fluid having conductivity σ and permittivity ε fills the space above a plane of electrodes to a depth Δ as shown above. The electrodes are excited so as to support a wave of potential that travels in the $\hat{z}$ direction with amplitude $\hat{V}$, temporal frequency ω and spatial wavenumber k. This electrical excitation pumps the fluid so that its interface with the free space at $x = \Delta$ travels with velocity U in the $\hat{z}$ direction. Assume that the system operates in the sinusoidal steady state so that the electric potential in the fluid and free space regions take the form

$$\Phi_{Fluid} = \text{Real}\left\{\left(\hat{\phi}_A \frac{\sinh(kx)}{\sinh(k\Delta)} - \hat{\phi}_B \frac{\sinh(k(x-\Delta))}{\sinh(k\Delta)}\right)e^{j(\omega t - kz)}\right\}$$

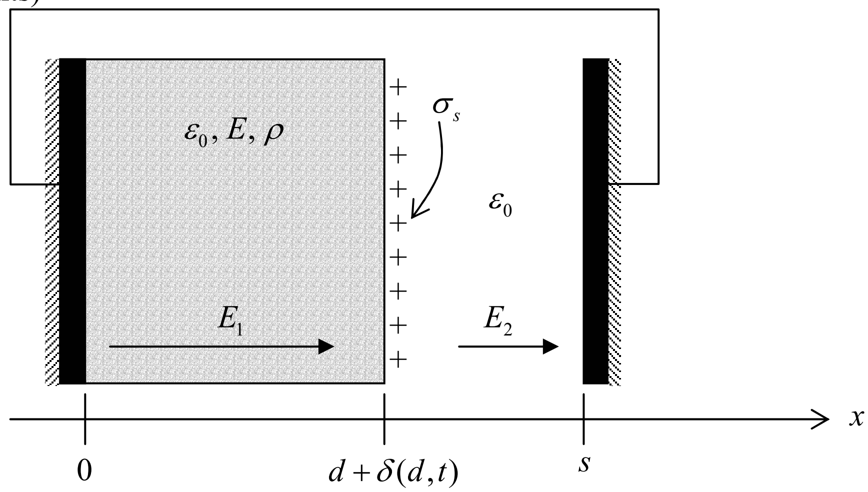
$$\Phi_{Free\ space} = \text{Real}\left\{\hat{\phi}_C e^{-k(x-\Delta)} e^{j(\omega t - kz)}\right\}$$

and the free surface charge density at the fluid-to-free-space interface at $x = \Delta$ takes the form

$$\rho_s = \text{Real}\left\{\hat{\rho}_s e^{j(\omega t - kz)}\right\}$$

a) Find the electric field in the fluid, and in the free space region above the fluid, in terms of $\hat{\phi}_A, \hat{\phi}_B$ and $\hat{\phi}_C$.
b) Using the boundary conditions for an EQS system associated with Gauss' Law and an irrotational $\overline{E}$ field, write three boundary conditions that relate $\hat{\phi}_A, \hat{\phi}_B, \hat{\phi}_C$ and the surface charge density $\hat{\rho}_S$ to each other, and the parameters of the fluid and the excitation.
c) Using the boundary condition for charge conservation at $x = \Delta$, write a fourth boundary condition that relates $\hat{\phi}_A, \hat{\phi}_B, \hat{\phi}_C$ and $\hat{\rho}_S$ to each other, and the parameters of the fluid and the excitation.
d) Combine the four boundary conditions found in Parts A and B to determine $\hat{\phi}_A, \hat{\phi}_B, \hat{\phi}_C$ and ρ in terms of the parameters of the fluid and the excitation.

Problem 3 (20 points)



An elastic dielectric with free space permittivity ε_0, mass density ρ, modulus of elasticity *E*, and unstretched length *d* is placed between short circuited electrodes of fixed spacing *s*. Both electrodes are fixed and cannot move. The fixed left electrode is glued to the elastic dielectric at *x=0* so that the elastic displacement at *x=0* is zero. There is an air gap between the elastic dielectric and the right electrode. On the interface between the elastic dielectric and air-gap is placed a constant value of surface charge with density σ_s Coulomb/m^2. The elastic dielectric can have small-signal displacement $\delta(x,t)$ and the small signal elastic displacement at *x=d* is $\delta(d,t)$. Note again, the elastic dielectric and the free space air-gap have the same dielectric permittivity ε_0. Neglect fringing field effects.

a) For a value of interfacial displacement $\delta(d,t)$, the electric fields E_1 and E_2 in the elastic dielectric and free space are of the form

$$E_1 = A(B + d + \delta(d,t))$$

$$E_2 = C(D + d + \delta(d,t))$$

where *A, B, C*, and *D* are constants. What are *A, B, C*, and *D*?

b) The electric force per unit area on the interface at $x = d + \delta(d,t)$ takes the form

$$\frac{F_e}{Area} = F + G\delta(d,t)$$

where *F* and *G* are constants. Using the results of part (a), determine *F* and *G*?

c) What is the steady state elastic displacement $\delta_{ss}(x)$?

d) Now assume that the system is slightly perturbed so that the elastic displacement is of the form

$$\delta(x,t) = \delta_{ss}(x) + \delta'(x,t)$$

Take the general form of $\delta'(x,t)$ to be

$$\delta'(x,t) = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$$

and find the spatial dependence for $\hat{\delta}(x)$.

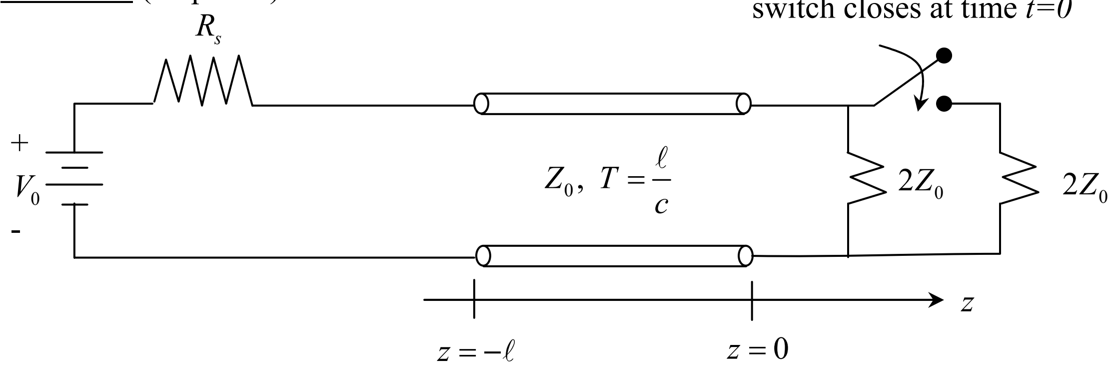
e) The system natural frequencies take the form

$$\tan(kd) = Hkd, \quad k = \omega\sqrt{\frac{\rho}{E}}$$

where *H* is a constant. What is *H*?

f) At what value of σ_s is the system first unstable?

Problem 4 (20 points)



A transmission line with characteristic impedance Z_0, length l and wave speed c has a load at $z=0$ of $2Z_0$ when the switch is open, and a load of Z_0 when the switch is closed. At $z=-\ell$, the transmission line is connected to a DC voltage V_0 and series source resistance R_s.

a) The switch at $z=0$ is open for $t<0$ and the voltage source has been connected to the transmission line for a long time so that all transient waves have died away and the transmission line voltage and current are in the DC steady state. What are the steady-state voltage and current on the transmission line for $t<0$?

b) With the transmission line in the DC steady state of part (a), the switch at $z=0$ is closed for all time $t>0$. The resulting voltage and current transient waves on the transmission line can be written as

$$v(z,t)=V_+(t-\frac{z}{c})+V_-(t+\frac{z}{c}) \qquad ; \qquad i(z,t)Z_0=V_+(t-\frac{z}{c})-V_-(t+\frac{z}{c})$$

What are $V_+(t-\frac{z}{c})$ and $V_-(t+\frac{z}{c})$ at times $t=0$ and $t=\infty$?

c) At $z=0$, with the switch closed for $t>0$, calculate

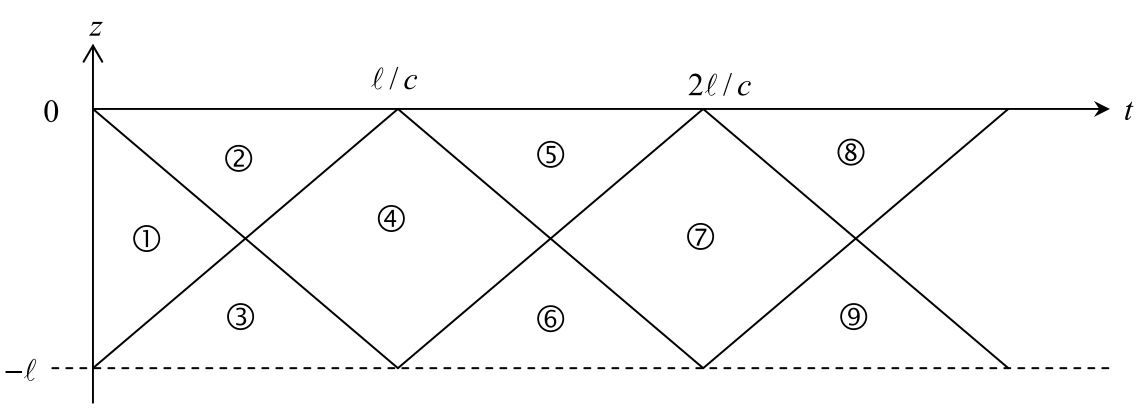
$$\left.\frac{V_-}{V_+}\right|_{z=0}$$

d) At $z=-\ell$ for $t>0$, the positive z directed wave is of the form

$$V_+\big|_{z=-\ell}=A+BV_-\big|_{z=-\ell}$$

What are A and B?

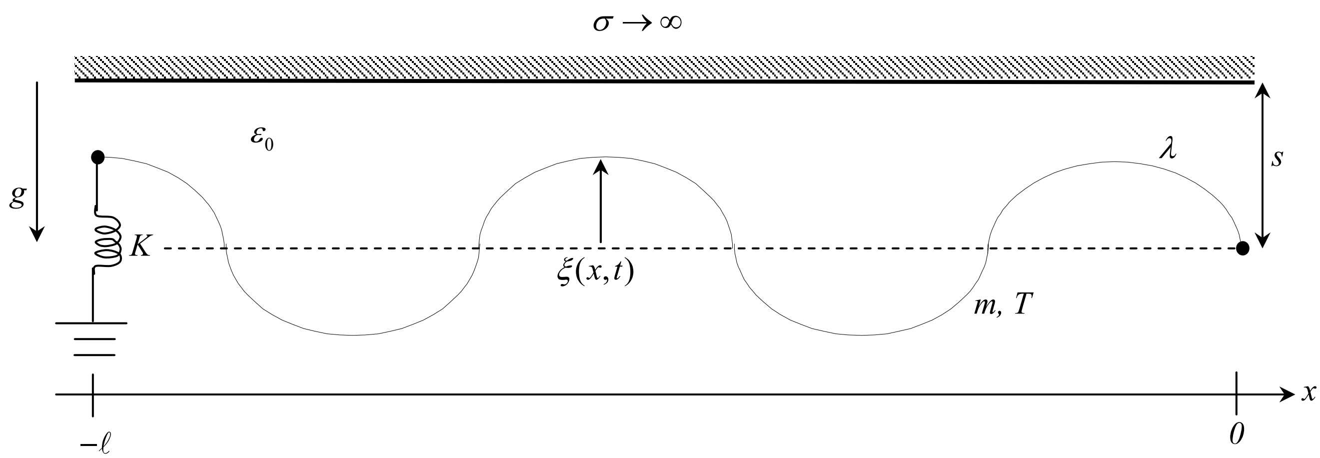
e) The wave trajectories in z-t space demarcate the solution regions as shown below.



Consider the case when $R_s=0$. What are V_+ and V_- in regions 1, 2, 3, 4, 5, 6, 7, 8, 9?

f) Give a labeled plot of $v(z=-\frac{\ell}{4},t)$ and $i(z=-\frac{\ell}{4},t)Z_0$ for all time $t\geq 0$.

Problem 5 (20 points)



A string of mass m per unit length and uniform tension T has a uniformly distributed line charge λ Coulombs/meter over its length ℓ. The charged string is placed a distance s below a perfectly conducting ground plane and is surrounded by free space. The string supports small signal displacements $\xi(x,t)$ and is fixed at $x=0$. At $x=-\ell$ the string is tied to a linear spring with spring constant K. The point where the string and spring are tied together can only move vertically. The spring exerts no force when $\xi(x=-\ell,t)=0$ so that the force exerted by the spring is $-K\xi(x=-\ell,t)$. Gravity acts downwards.

a) In the long wavelength limit, what is the electric force per unit length on the string to first order in $\xi(x,t)$ when $\xi(x,t) \ll s$?
b) For what value of λ will the string have an equilibrium with $\xi(x,t)=0$?
c) For small signal wave solutions of the form

   $$\xi(x,t) = \mathrm{Re}\left[\hat{\xi} e^{j(\omega t - kx)}\right]$$

   what is the $\omega - k$ dispersion relation?
d) Applying the boundary conditions at $x=0$ and $x=-\ell$, the allowed values of k can be obtained from the transcendental relationship of the form

   $$\tan(k\ell) = Ck\ell$$

   where C is a constant. What is C?
e) If the spring constant is zero, $K=0$, what are the solutions for wavenumber k from part (d)?
f) For the conditions of part (e), what is the maximum string mass per unit length m that can be stably supported with $\xi(x,t)=0$?