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6.641 Electromagnetic Fields, Forces, and Motion, Spring 2005

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Massachusetts Institute of Technology  
 Department of Electrical Engineering and Computer Science  
 6.641 Electromagnetic Fields, Forces, and Motion

Problem Set #4  
 Spring Term 2005

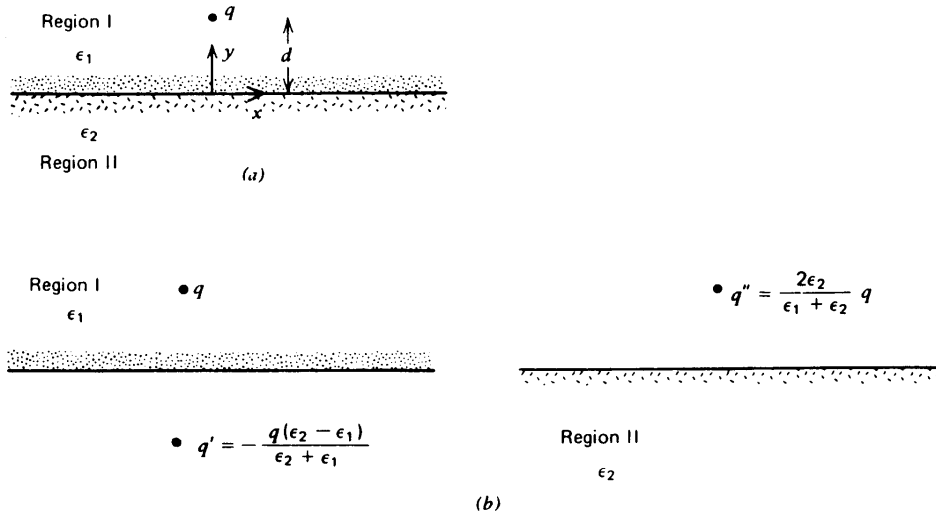
Issued: 2/24/05  
 Due: 3/3/05

Suggested Reading Assignment: Zahn- 3.1-3.6, 5.6

Suggested Video Viewing: H/M Demos 7.7.1 and Supplement, 8.6.1  
[\[http://web.mit.edu/6.013\\_book/www/VideoDemo.html\]](http://web.mit.edu/6.013_book/www/VideoDemo.html) and Demo: 9.4.1

Quiz 1 – Wednesday, March 9, 2005, 7:30-9:30PM The 6.641 Formula Sheet (attached) will be supplied. You are also allowed to bring one 8 1/2" x 11" sheet of notes (both sides) that you prepare. The exam covers all material up to and including Problem Set #4 material.

Problem 4.1



Zahn diagrams from *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, Robert E. Krieger Publishing Company, 1987. Used with permission.

(a) A point charge  $q$  above a flat dielectric boundary requires different sets of image charges to solve for the fields in each region. (b) The field in region I is due to the original charge and the image charge  $q'$  while the field in region II is due only to image charge  $q''$ .

A point charge  $q$  is within a region of permittivity  $\epsilon_1$  and is a distance  $d$  above a planar boundary separating region I from region II, which has a permittivity  $\epsilon_2$ . There is no unpaired (free) surface charge on the interface at  $y=0$ . The tangential component of  $\vec{E}$  and the normal component of  $\vec{D}$  must be continuous across the interface.

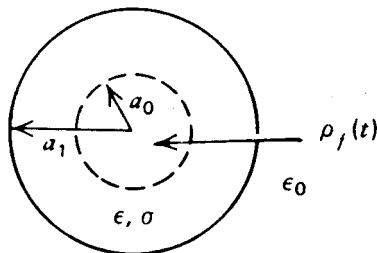
Let us try to use the method of images by placing an image charge  $q'$  at  $y=-d$  so that the solution in region I is due to this image charge plus the original point charge  $q$ . The solution for the field in region II will be due to an image charge  $q''$  at  $y=d$ , the position of the original point charge. Note that the appropriate image charge is always outside the region where the solution is desired. At this point we do not know if it is possible to satisfy the boundary conditions with these image charges, but we will try to find values of  $q'$  and  $q''$  to do so.

- a) What are the potential and electric field distributions in region I ( $y>0$ ) and region II ( $y<0$ ) in terms of  $q$ ,  $q'$ , and  $q''$ ?
- b) What equations relate  $q$ ,  $q'$ , and  $q''$  to satisfy continuity of tangential  $\vec{E}$  and normal  $\vec{D}$ ? Solve for  $q'$  and  $q''$  in terms of  $q$ ,  $\epsilon_1$  and  $\epsilon_2$ .
- c) What is the force on the point charge  $q$ ?

### Problem 4.2

An infinitely long cylinder of radius  $a_1$ , permittivity  $\epsilon$ , and conductivity  $\sigma$  is nonuniformly charged at  $t=0$ :

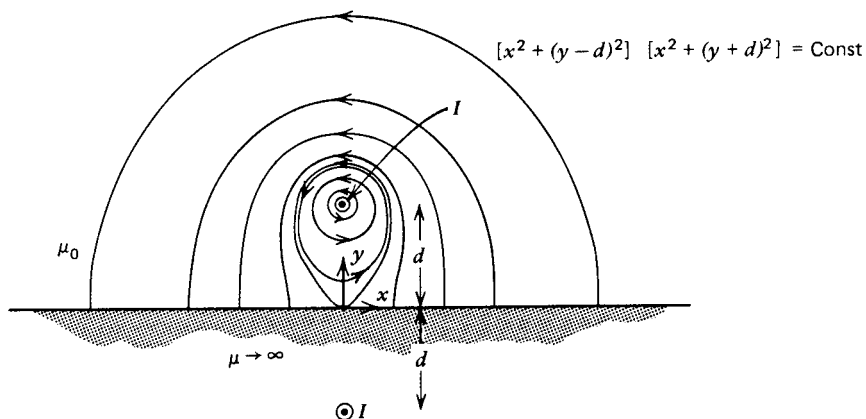
$$\rho_f(t=0) = \begin{cases} \frac{\rho_0 r}{a_0} & 0 < r < a_0 \\ 0 & r > a_0 \end{cases}$$



Zahn diagrams from *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, Robert E. Krieger Publishing Company, 1987. Used with permission.

What is the time dependence of the electric field everywhere and the free surface charge density at  $r=a_1$  as a function of time?

### Problem 4.3



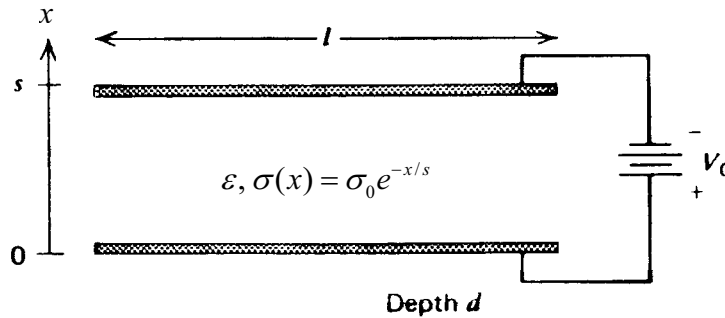
Zahn diagrams from *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, Robert E. Krieger Publishing Company, 1987. Used with permission.

A line current  $I$  of infinite extent in the  $z$  direction is at a distance  $d$  above an infinitely permeable material as shown.

- What is the boundary condition on the magnetic field at  $y=0$ ?
- Use the method of images to satisfy the boundary condition of (a) and find the magnetic vector potential for  $y>0$ . Hint: The solution is very similar to that of Homework Problem 3.5.
- What is the magnetic field for  $y>0$ ?
- Derive the equation of the magnetic field lines for  $y>0$ .
- What is the force per unit length on the line current at  $y=d$ ?

Problem 4.4

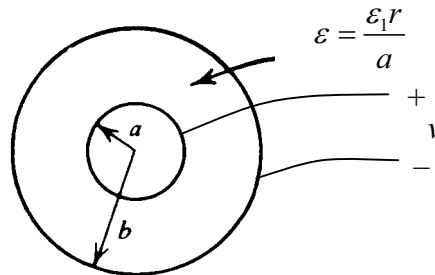
A pair of parallel plate electrodes at voltage difference  $V_0$  enclose an Ohmic material whose conductivity varies exponentially with position as  $\sigma(x) = \sigma_0 e^{-x/s}$ . The permittivity  $\epsilon$  of the material is a constant.



Zahn diagrams from *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, Robert E. Krieger Publishing Company, 1987. Used with permission.

- Find the electric field  $E_x(x)$  and the resistance between the electrodes.
- What are the volume and surface charge densities?
- What is the total volume charge in the system and how is it related to the surface charge on the electrodes?

Problem 4.5



Zahn diagrams from *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, Robert E. Krieger Publishing Company, 1987. Used with permission.

Concentric spherical electrodes with respective radii  $a$  and  $b$  enclose an Ohmic material whose permittivity varies with radius as  $\epsilon(r) = \frac{\epsilon_1 r}{a}$ . A voltage  $v$  is applied across the spherical electrodes. There is no volume charge in the dielectric.

- What are the electric field and potential distributions for  $a < r < b$ ?
- What are the surface charge densities at  $r = a$  and  $r = b$ ?
- What is the capacitance?

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6.641 FORMULA SHEET

**1. DIFFERENTIAL OPERATORS IN CYLINDRICAL AND SPHERICAL COORDINATES**

If  $r$ ,  $\phi$ , and  $z$  are circular cylindrical coordinates and  $\hat{i}_r$ ,  $\hat{i}_\phi$ , and  $\hat{i}_z$  are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = \text{grad} U = \hat{i}_r \frac{\partial U}{\partial r} + \hat{i}_\phi \frac{1}{r} \frac{\partial U}{\partial \phi} + \hat{i}_z \frac{\partial U}{\partial z}$$

$$\nabla \cdot \vec{A} = \text{div} \vec{A} = \frac{1}{r} \frac{\partial (r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = \text{curl} \vec{A} = \hat{i}_r \left( \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{i}_\phi \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{i}_z \left( \frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \phi} \right)$$

$$\nabla^2 U = \text{div grad} U = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

If  $r$ ,  $\theta$ , and  $\phi$  are spherical coordinates and  $\hat{i}_r$ ,  $\hat{i}_\theta$ , and  $\hat{i}_\phi$  are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = \text{grad} U = \hat{i}_r \frac{\partial U}{\partial r} + \hat{i}_\theta \frac{1}{r} \frac{\partial U}{\partial \theta} + \hat{i}_\phi \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi}$$

$$\nabla \cdot \vec{A} = \text{div} \vec{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \vec{A} = \text{curl} \vec{A} = \hat{i}_r \left( \frac{1}{r \sin \theta} \frac{\partial (A_\phi \sin \theta)}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi} \right) + \hat{i}_\theta \left( \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} \right) + \hat{i}_\phi \left( \frac{1}{r} \frac{\partial (r A_\theta)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right)$$

$$\nabla^2 U = \text{div grad} U = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2}$$

**2. SOLUTIONS OF LAPLACE'S EQUATIONS**

**A. Rectangular coordinates, two dimensions (independent of z):**

$$\Phi = e^{kx} (A_1 \sin ky + A_2 \cos ky) + e^{-kx} (B_1 \sin ky + B_2 \cos ky)$$

(or replace  $e^{kx}$  and  $e^{-kx}$  by  $\sinh kx$  and  $\cosh kx$ ).

$$\Phi = Axy + Bx + Cy + D; (k = 0)$$

**B. Cylindrical coordinates, two dimensions (independent of z):**

$$\Phi = r^n (A_1 \sin n\phi + A_2 \cos n\phi) + r^{-n} (B_1 \sin n\phi + B_2 \cos n\phi)$$

$$\Phi = \ln \frac{R}{r} (A_1 \phi + A_2) + B_1 \phi + B_2; (n = 0)$$

**C. Spherical coordinates, two dimensions (independent of  $\phi$ ):**

$$\Phi = Ar \cos \theta + \frac{B}{r^2} \cos \theta + \frac{C}{r} + D$$