LECTURE 2: One-Dimensional 'Traveling Waves'

Main Points

- Exponential and sine-wave solutions to the one-dimensional wave equation.
- The distributed compressibility and mass in acoustic plane waves are analogous with the distributed capacitance and inductance in electrical transmission lines.
- Traveling waves vary in both space and time.
- Interactions of waves with structures of different impedance that are of significant size compared to a wavelength produce reflected waves.
- The magnitude of the reflection depends on the relative impedance of the object and the media.

1. The One-Dimensional Wave Equation for Plane Waves:

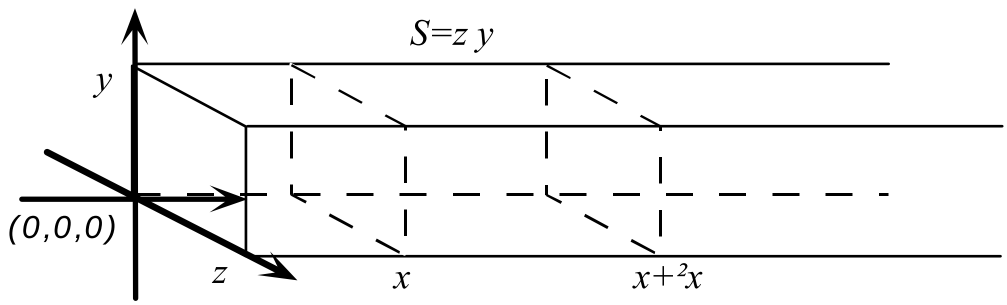


Figure 2.1: A long duct of height *y*, width *z* and undetermined length. Our derivation of the wave equation is based on a section of duct described by the interval *x* to $x+\Delta x$.

We saw in Lecture 1 that we can characterize the propagation of plane waves fairly simply, if we make some generally reasonable assumptions:

a. the forces related to the viscosity of air are negligible, and

b. the rapid variations in pressure associated with sound don't allow heat transfer within the medium or to the surround (the adiabatic condition),

c. the sound induced variations in the scalars $p(x,t)$, $\rho(x,t)$ and $T(x,t)$ are small compared to their static values.

d. the sound induced particle velocity $v_x(x,t)$ is small compared to the propagation velocity.

These assumptions together with considerations of Newton's second law, conservation of mass and consideration of the adiabatic compressibility of air lead to **lossless** acoustic equations (consistent with a and b above) in which the distributed mass (the density ρ_0) and distributed compliance (the bulk modulus B_A) of the air completely determine the relationship between v_x (the magnitude of the x component of the particle velocity) and p (the sound pressure) at any position (x) and time (t) in a one dimensional system like Figure 2.1.

Newtons 2nd Law:

$$\frac{\partial p(x,t)}{\partial x} = -\rho_0 \frac{\partial v_x(x,t)}{\partial t} \qquad (1.2.7)$$

Conservation of Mass-Compressibility Relationship:

$$\frac{\partial v_x(x,t)}{\partial x} = -\frac{1}{B_A}\frac{\partial p(x,t)}{\partial t} \,. \qquad (1.2.20)$$

The Wave-Equation for Sound Pressure in a Plane Wave:

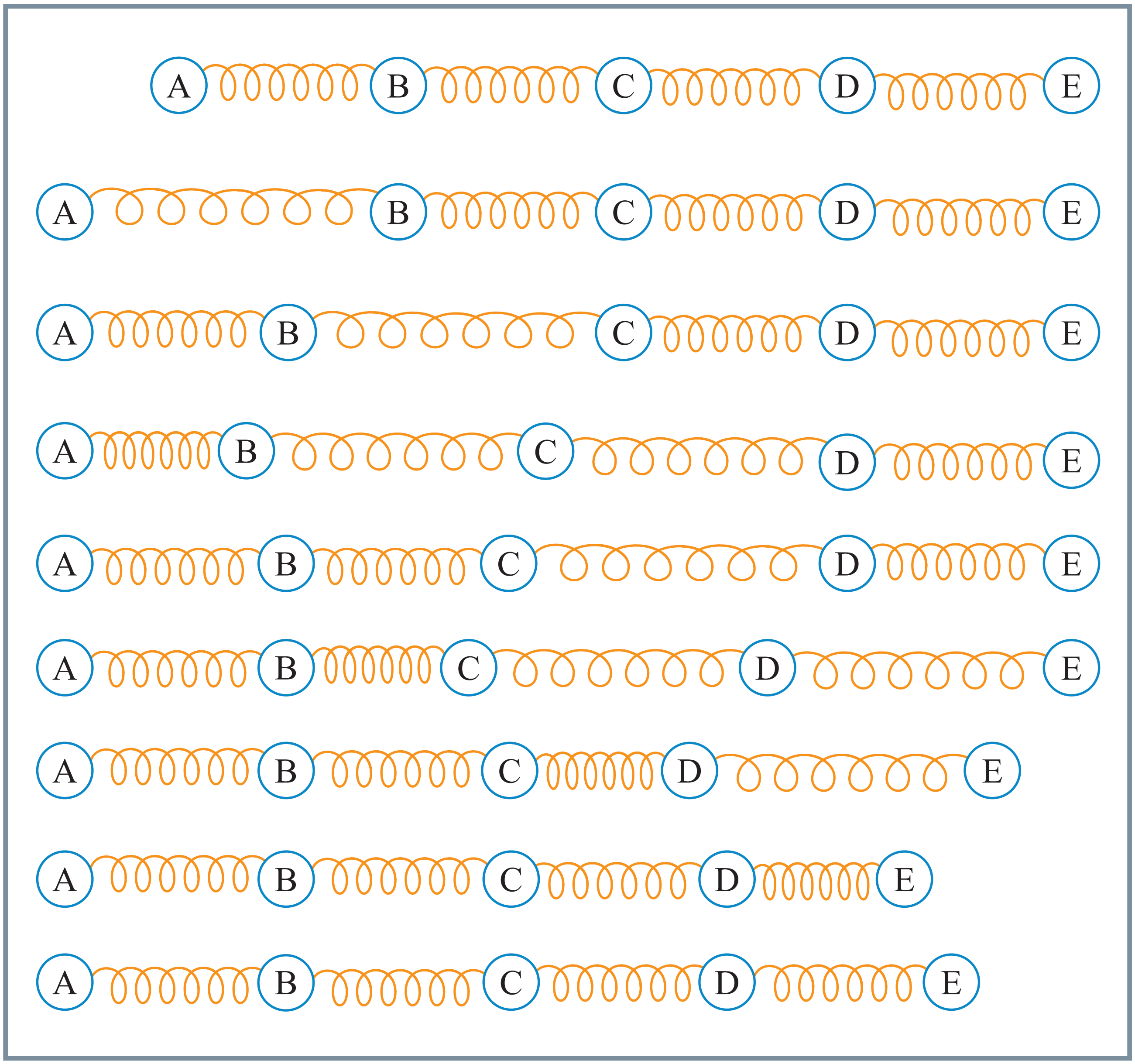
$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{\rho_0}{B_A}\frac{\partial^2 p(x,t)}{\partial t^2}, \text{ where } \frac{\rho_0}{B_A} = \frac{1}{c^2}. \quad (1.2.21)$$

We can write a 'matching' equation to describe the variation in particle velocity as a function of time by taking the partial of both sides of 1.2.7 with respect to time, and substituting 1.2.20 into the left-hand side of the result:

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{\rho_0}{B_A}\frac{\partial^2 v(x,t)}{\partial t^2}. \quad (2.1)$$

2. Wave propagation.

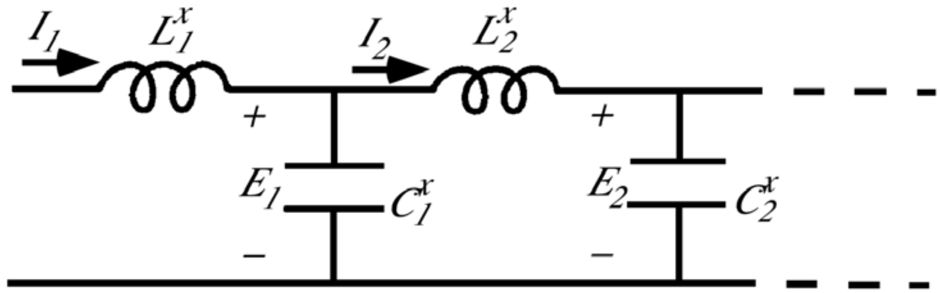
The propagation of sound as described by the wave equation can be understood by a 'distributed' series of 'lumped' masses and compliance (spring-like) elements. The following figure is a modification of Denes and Pinsons's Figure 3, in which a perturbation in a string of springs and masses causes a propagated wave of force and motion, (modified from Denes & Pinson "The Speech Chain", WH Freeman 1993).



In the top row the balls and springs are at rest
In the second row, ball A is displaced to the left, stretching the AB spring
In the third row, ball B has moved toward A, compressing AB and stretching BC
In the fourth row Ball B moves even closer to A compressing AB past its rest position
In the fifth row Ball B moves back to the left and settles into its new rest position, etc.

3. Similarity to Transmission Line Equations

We can use the acoustic analog of what are known as transmission line equations to describe sound in a one-dimensional plane wave. In such a system, the wave is described in terms of the interaction of a distributed series of lumped electrical capacitors and inductances. In this analogy the voltage *e(t,x)* is the analog of pressure, the current *i(t,x)* is analogous to the one-dimensional particle velocity, the inductances are analogous to acoustic inertances per unit distance, and the capacitors are analogous to acoustic compliance per unit distance.



L^x = is the electrical inductance per unit length (the electrical analog of density ρ_0),

C^x = the electrical capacitance per unit length (the analog of the compressibility of air $1/B_A$),

I = a complex amplitude that describes the current (the analog of particle velocity), and

E = a complex amplitude that describes the voltage (the analog of sound pressure).

The inductance/length:
$$\frac{\partial e(x,t)}{\partial x} = -L^x \frac{\partial i(x,t)}{\partial t} \tag{2.2}$$

The compliance per length:
$$\frac{\partial i(x,t)}{\partial x} = -C^x \frac{\partial e(x,t)}{\partial t} \quad . \tag{2.3}$$

The variations in voltage and current in time and space can be described by wave equations:

$$\frac{\partial^2 e(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 e(x,t)}{\partial t^2}, \text{ and } \frac{\partial^2 i(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 i(x,t)}{\partial t^2}, \text{ where: } c = \sqrt{\frac{1}{L^x C^x}} \quad . \tag{2.4,5 \&6}$$

We can also define an electrical impedance in the transmission line where

$$Z = \sqrt{L^x / C^x} \quad . \tag{2.7}$$

All of these equations are analogous to the equations we derived for plane-wave propagation of sound, where:

$$\begin{aligned} e(t) &\rightarrow p(t) \\ i(t) &\rightarrow v_x(t) \\ L^x &\rightarrow \rho_0 \\ C^x &\rightarrow 1/B_A \end{aligned} \quad .$$

4. Solutions to the Wave Equation

A general solution for one-dimensional plane-wave propagation describes the pressure and particle velocity at any time and as the sum of two **traveling waves**, one moving in the positive x direction and the other negative going:

$$p(t,x) = f^{+}(t - x/c) + f^{-}(t + x/c) \quad , \tag{2.8}$$

and

$$v_x(t,x) = \frac{1}{z_0}\left[f^{+}(t - x/c) - f^{-}(t + x/c)\right], \tag{2.9}$$

where the argument to the wave functions (f^{+} and f^{-}) is a time (t-x/c) determined by the absolute time t and the time needed to travel to x, i.e. x/c.

You should note:

– v_x and p in each wave are related by the characteristic impedance of the medium z_0.
– The two scalar pressure terms add.
– Because of a difference in direction, the two velocity terms subtract.
– Because of the difference in the signs of the second terms, $v_x(t,x)$ and $p(t,x)$ need not be proportional.

An alternative form of this solution can be given in terms of absolute position and the distance propagated in a given time:

$$p(t,x) = g^{+}(x - ct) + g^{-}(x + ct), \quad \text{and} \tag{2.10}$$

$$v_x(t,x) = \frac{1}{z_0}\left(g^{+}(x - ct) - g^{-}(x + ct)\right) , \tag{2.11}$$

Equations 2.8&9 define $v_x(x,t)$ and $p(x,t)$ in terms of two functions (f^{+} and f^{-}) that depend on the sound source and the boundary conditions at the two ends of our one-dimensional system. In the case of a completely open space the sound produced by a source propagates along its one dimensional axis as a forward traveling wave, and there is no backward traveling wave:

$$p(t,x) = f^{+}(t - x/c), \text{ and } v_x(t,x) = \frac{1}{z_0}\left(f^{+}(t - x/c)\right). \tag{2.12}$$

Why are these Forward Traveling waves?

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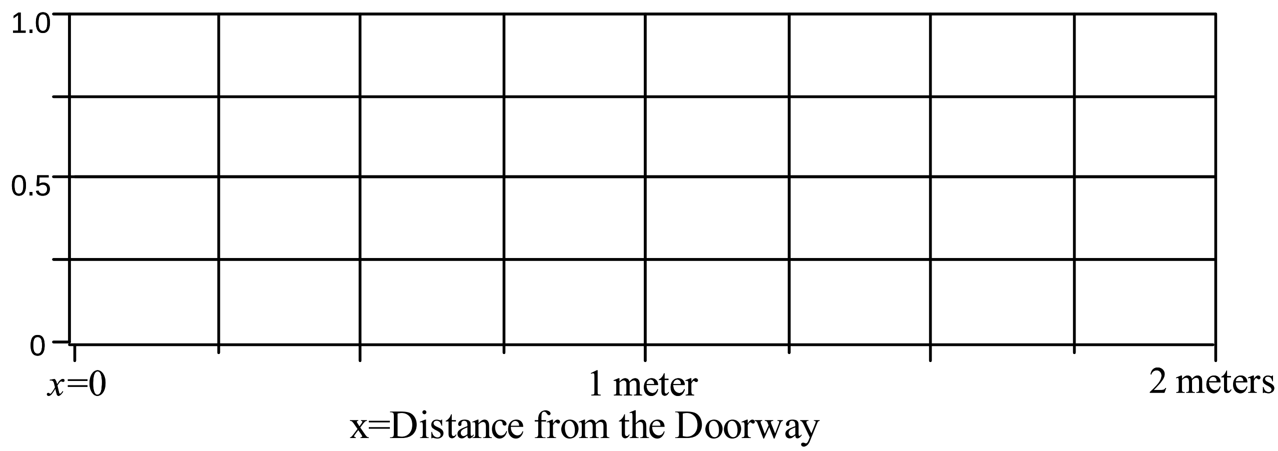
Think about how sound travels. Assume (1) the sound pressure in this room is zero at all negative times and (2) at t=0 we generate a one dimensional plane-wave pulse of pressure of 1 Pa peak. To be precise:

$$\begin{aligned} p(t,x) &= 0 \text{ when } t < 0; \\ p(0,0) &= 1; \\ p(t,0) &= 0 \text{ at } t > 0 \end{aligned} \qquad (2.13)$$

These boundary conditions when applied to Equation 2.12 suggest that the we can define the one dimensional forward traveling wave: as:

$$\begin{aligned} &p(t,x) = f^{+}(\zeta); \text{ where } \zeta = (t - x/c) \\ &\text{with } f^{+}(\zeta) = 0 \text{ for } \zeta < 0, \text{ and } \zeta > 0 \\ &\text{and } f^{+}(\zeta) = 1 \text{ for } \zeta = 0\,. \end{aligned} \qquad (2.14)$$

How does pressure vary within the room at t=0, t=1 ms, t=3 ms, t=5 ms ?



The pulse propagates as a "wave front" of the traveling wave.

5. Sinusoidal Traveling Waves:

A sinusoidal steady state solution for the wave equation also depends on the summation of a forward and backward going traveling wave

$$p(t,x) = \text{Real}\left\{\underline{P}^{+} e^{j\omega(t-x/c)} + \underline{P}^{-} e^{j\omega(t+x/c)}\right\} \tag{2.15}$$

$$v_x(t,x) = \text{Real}\left\{\frac{1}{z_0}\left(\underline{P}^{+} e^{j\omega(t-x/c)} - \underline{P}^{-} e^{j\omega(t+x/c)}\right)\right\}, \tag{2.16}$$

> For those of you still not comfortable with the exponential notation, remember that Eqn. 2.15 is equivalent to:
>
> $$p(t,x) = \left|\underline{P}^{+}\right| \cos\left(\omega(t - x/c) + \angle \underline{P}^{+}\right) + \left|\underline{P}^{-}\right| \cos\left(\omega(t + x/c) + \angle \underline{P}^{-}\right).$$
>
> How does this equivalence come about?

Equations 2.15 and 2.16 can also be written in terms of the variable $k = \omega/c = 2\pi/\lambda$, where k has units of radians per meter and is sometimes called the wave number, length constant or spatial frequency :

$$p(t,x) = \text{Real}\left\{\underline{P}^{+} e^{j(\omega t - kx)} + \underline{P}^{-} e^{j(\omega t + kx)}\right\}, \text{ and} \tag{2.17}$$

$$v_x(t,x) = \text{Real}\left\{\frac{1}{z_0}\left(\underline{P}^{+} e^{j(\omega t - kx)} - \underline{P}^{-} e^{j(\omega t + kx)}\right)\right\}. \tag{2.18}$$

Suppose the sound pressure source in one of the walls produces a steady-state sinusoidal variation in pressure with radial frequency $\omega = 2\pi$ *170 Hz :*

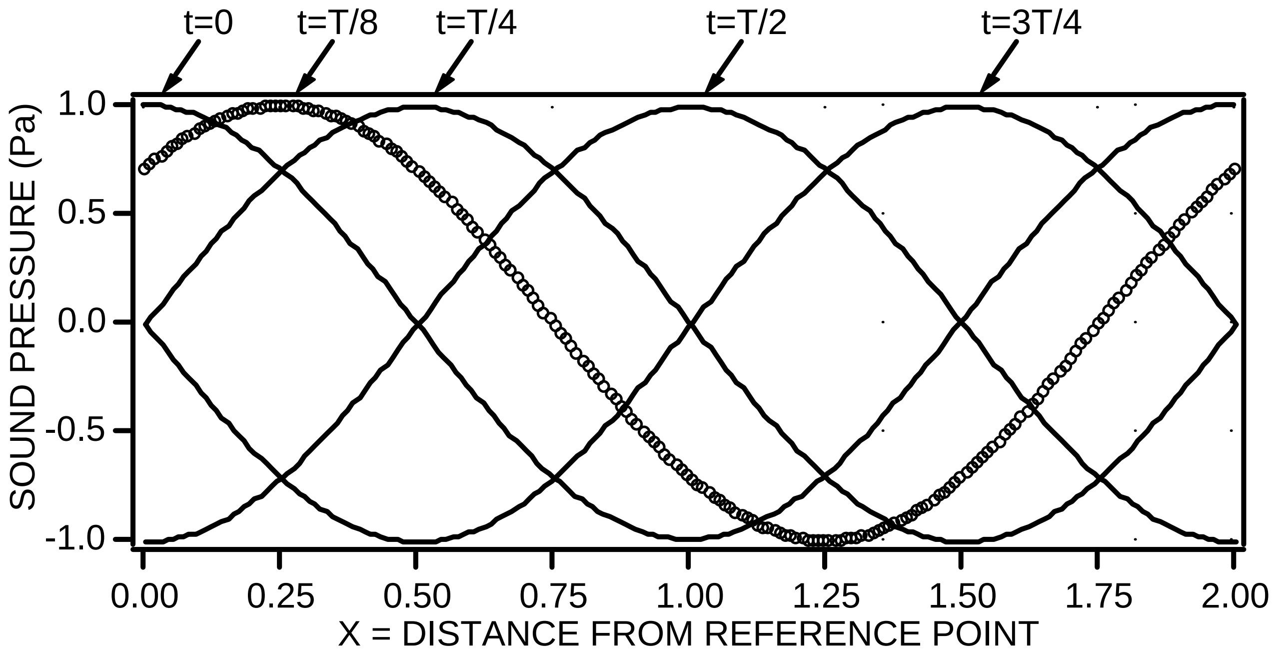
$$p(t,0) = \cos(\omega t) = \text{Real}\{e^{j\omega t}\}.$$

The sinusoid also *"travels"* across the room at a velocity of c, i.e.

$$p(t,x) = \cos(\omega\zeta); \text{ where } \zeta = (t - x/c).$$

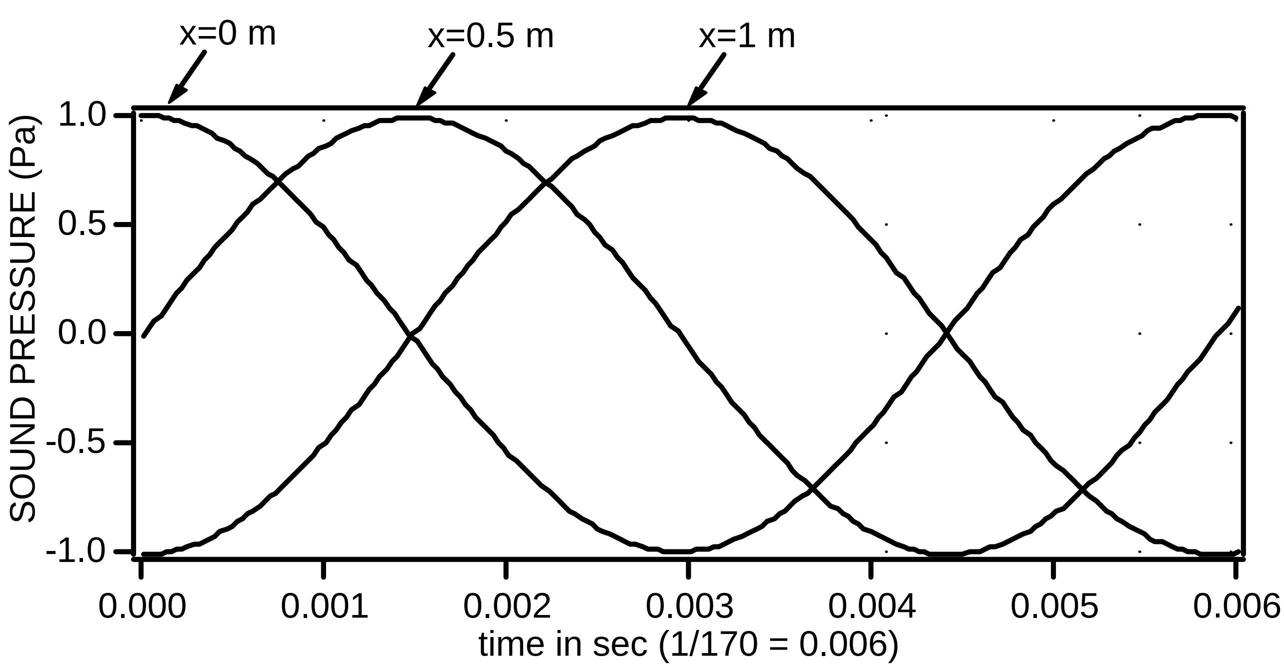
How does sound pressure vary across the room at time 0 and at fractions of a period later? (Hint: What is the wavelength of a sound of 170 Hz?)

$p(t,x)=\cos(\omega\zeta)$; where $\zeta=(t - x/c)$.



As time progresses, the "wave-front" (here defined as the location of maximum pressure) travels across the room with a velocity *c*.

Now suppose we place a microphone at various locations in the room. How does the sound pressure vary with time at $x = 0$, x=0.5 meters, x=1 meters, and x=2 meters?



0ne-dimensional wave propagation depends on both time and space. The events that occur in the present at location $x=0$, predict the events that will occur further away from the source at a later time.

6. The separation of time and space dependence.

The time and space dependence of traveling waves can be separated from each other. In cases where the temporal dependence of the wave is well defined, such a separation allows us to concentrate on the spatial dependence. In the case of the sinusoidal steady state:

$$p(t,x) = \text{Real}\left\{\underline{P}^{+} e^{j(\omega t - kx)} + \underline{P}^{-} e^{j(\omega t + kx)}\right\} \tag{2.19}$$

we can factor out $e^{j\omega t}$,

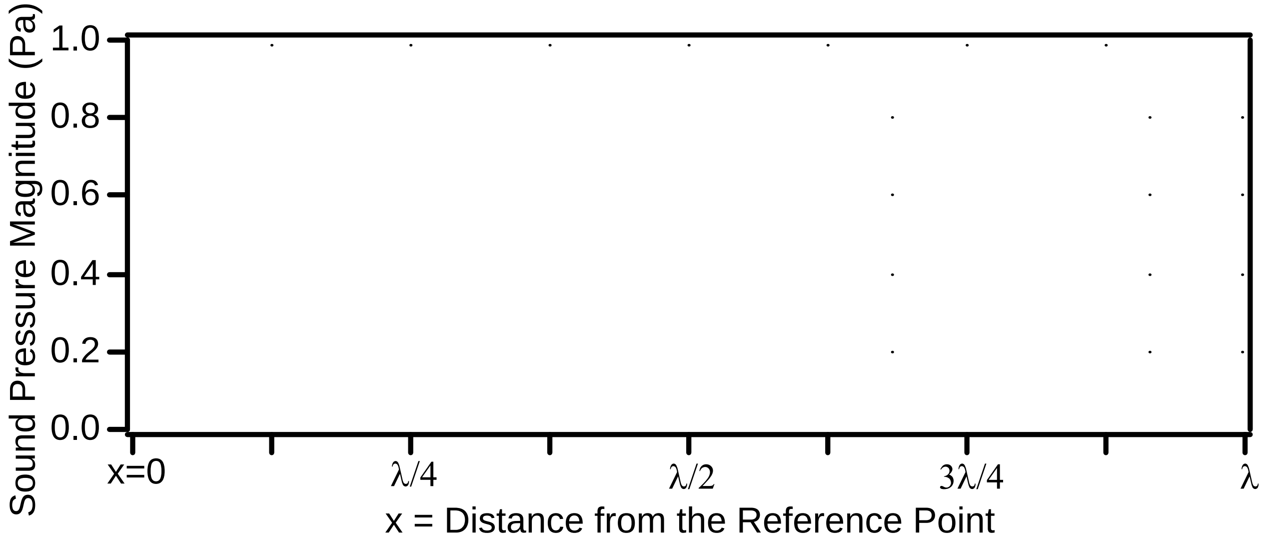
$$p(t,x) = \text{Real}\left\{e^{j\omega t}\,\underline{P}(x)\right\} \textit{ where}$$

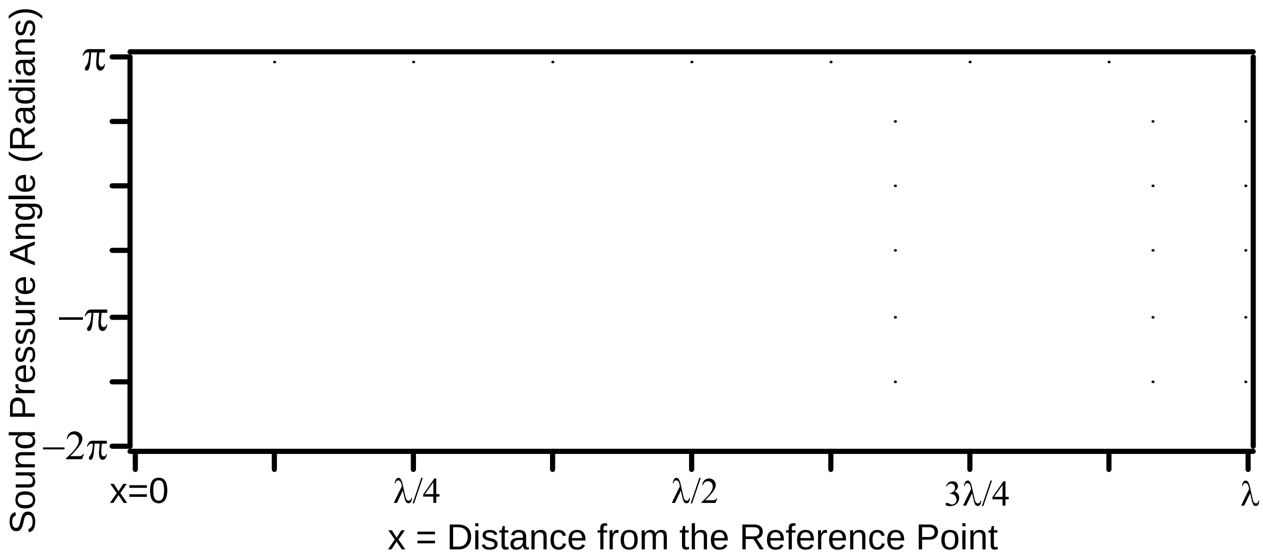
$$\underline{P}(x) = \left(\underline{P}^{+} e^{-jkx} + \underline{P}^{-} e^{jkx}\right) \tag{2.20}$$

In a wide open environment with no reflection, we can define the spatial dependence of a forward traveling plane wave, as

$$\underline{P}(x) = \left(\underline{P}^{+} e^{-jkx}\right) . \tag{2.21}$$

If $\underline{P}^{+}=1$, how do the magnitude and angle of $\underline{P}(x)$ vary in space?


Sound Pressure Magnitude (Pa)
1.0
0.8
0.6
0.4
0.2
0.0
x=0
λ/4
λ/2
3λ/4
λ
x = Distance from the Reference Point


Sound Pressure Angle (Radians)
π
–π
–2π
x=0
λ/4
λ/2
3λ/4
λ
x = Distance from the Reference Point

7. Reflections at Rigid Boundaries

Suppose our propagating plane wave hits a rigid wall placed orthogonally to the direction of propagation, where the wall dimensions are much larger than the wavelength. The interaction will produce a reflected wave that appears as a backward traveling wave in a one-dimensional system.

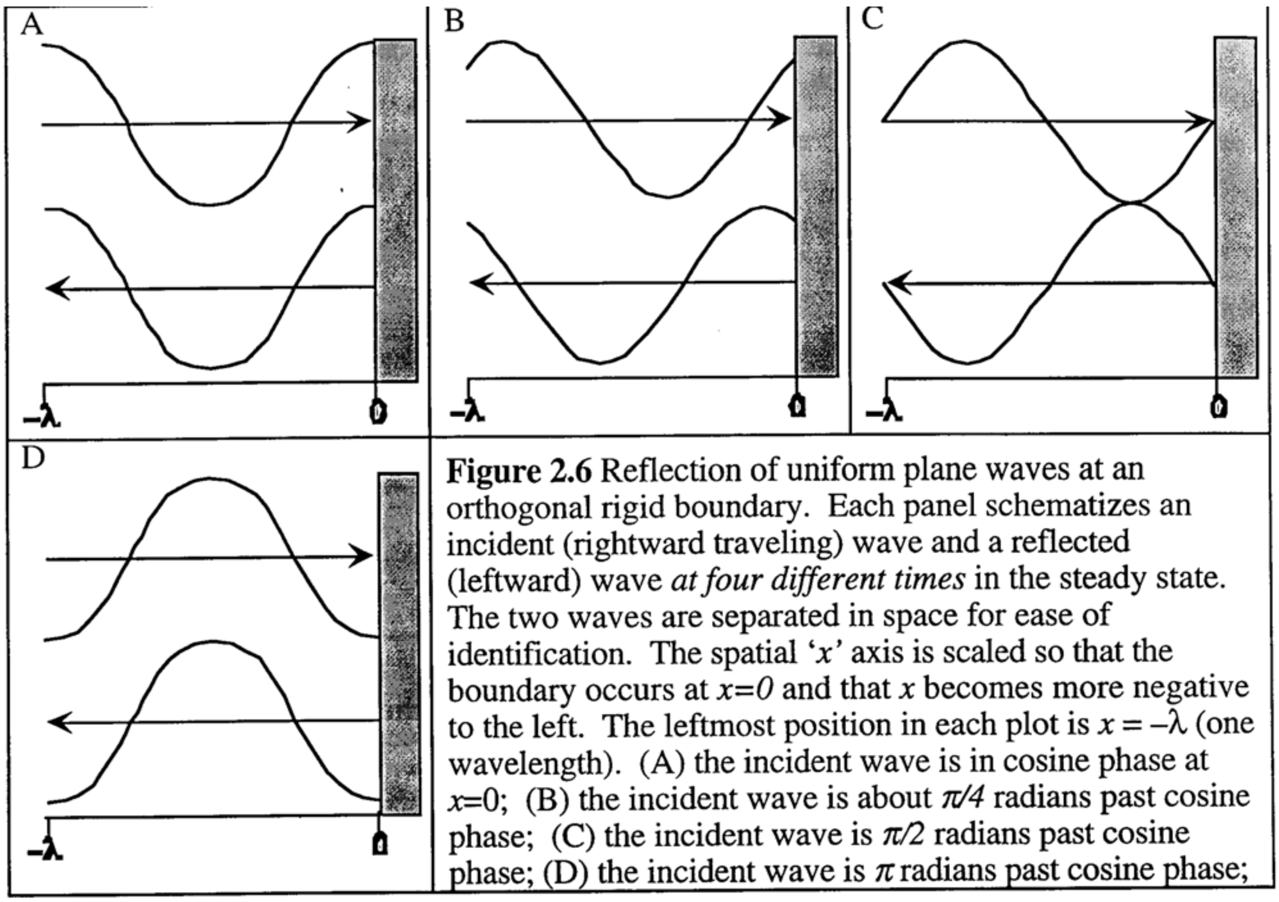


Figure 2.6 Reflection of uniform plane waves at an orthogonal rigid boundary. Each panel schematizes an incident (rightward traveling) wave and a reflected (leftward) wave *at four different times* in the steady state. The two waves are separated in space for ease of identification. The spatial '*x*' axis is scaled so that the boundary occurs at $x=0$ and that x becomes more negative to the left. The leftmost position in each plot is $x = -\lambda$ (one wavelength). (A) the incident wave is in cosine phase at $x=0$; (B) the incident wave is about $\pi/4$ radians past cosine phase; (C) the incident wave is $\pi/2$ radians past cosine phase; (D) the incident wave is π radians past cosine phase;

At the rigid boundary, the reflected wave acts as a continuation of the original wave, but its direction is altered. *In the steady state,* the sound pressure at each location is the sum of the two waves:

$$p(t,x) = \mathrm{Re}\left\{ \underline{P}^{+} e^{(\omega t - kx)} + \underline{P}^{-} e^{(\omega t + kx)} + \right\}$$

In the case of rigid boundary reflection in a one dimensional system:

(1) The amplitude and angle of the incident and reflected waves are equal $\underline{P}^{+}|=|\underline{P}^{-}$.
(2) The value of the incident and reflected pressure at the boundary is equal at all times
 $p^{+}(t,0)=p^{-}(t,0)$.
(3) The two waves always cancel at $n\lambda/4$, (n=1, 3, 5, …) distance from the wall.
(4) The sum of the two waves has a magnitude of $2|\underline{P}^{+}|$ at distances $m\lambda/2$ (m=0, 1, 2, …) from the wall.
(5) At times when the incident wave is in ±sine phase at the wall, the summed pressure is
 0 everywhere.

8. The Spatial Dependence of the Total Sound Pressure in Rigid Wall Reflection

Earlier, we described the total pressure at any location and time in terms of the sum of the forward and backward going waves:

$$p(t,x) = \text{Real}\left\{\underline{P}^+ e^{j(\omega t - kx)} + \underline{P}^- e^{j(\omega t + kx)}\right\} \tag{2.22}$$

We also separated out the temporal and location dependence, i.e.

$$p(t,x) = \text{Real}\left\{e^{j\omega t}\,\underline{P}(x)\right\}$$

where:

$$\underline{P}(x) = \underline{P}^+ e^{-jkx} + \underline{P}^- e^{jkx} \tag{2.23}$$

In the case of a forward traveling wave with a rigid boundary at *x*=0 where $\underline{P}^+ = \underline{P}^-$, as in Figure 2.6, Equation 2.23 simplifies via Euler's equations to

$$\underline{P}(x) = 2\underline{P}^+ \cos(kx), \tag{2.24}$$

Note that Equation 2.24:

(a) Is dependent on *x*, ω ($k=\omega/c$) but independent of *t*, this is a *standing wave.*

(b) The sound pressure at the rigid boundary (*x=0)* is twice the amplitude of the traveling waves $\underline{P}(0) = 2\underline{P}^+$.

(c) When $x=-\lambda/4$; $kx=\pi/2$ and $\underline{P}(-\lambda/4) = 0$; this zero is repeated at $x=-3\lambda/4$, $-5\lambda/4$, $-7\lambda/4$ …

(d) $\angle\underline{P}(x) = \angle\underline{P}^+$ and is invariant in space.

9. The Spatial Dependence of the Specific Acoustic Impedance in Rigid Wall Reflection

Rigid-walled reflection, where the angle of incidence is 90° relative to the boundary, also produces standing waves in particle velocity.

We can define $\underline{V}_x(x)$ starting from Equation 2.16:

$$v_x(t,x) = \text{Real}\left\{\frac{1}{z_0}\left(\underline{P}^+ e^{j\omega(t-x/c)} - \underline{P}^- e^{j\omega(t+x/c)}\right)\right\} \tag{2.16}$$

where:

$$v_x(t,x) = \text{Real}\left\{\frac{1}{z_0} e^{j\omega t}\,\underline{P}(x)\right\}, \textit{with}$$

$$\underline{V}_x(x) = \frac{1}{z_0}\left(\underline{P}^+ e^{-jkx} - \underline{P}^- e^{jkx}\right) = -2j\frac{\underline{P}^+}{z_0}\sin(kx) \tag{2.25}$$

Note that for $x < 0$; $\angle\underline{V}_x(x) = (\left(\pi/2 + \angle\underline{P}^+\right)$, has a magnitude of 0 at *x=0,* and has a magnitude maximum of $2\,|\underline{P}^+|/z_0$ at $x=-\lambda/4$, $-3\lambda/4$, $-5\lambda/4$, $-7\lambda/4$ …

The ratio of $\underline{P}(x)$ and $\underline{V}_x(x)$ defines the spatially varying specific acoustic impedance $\underline{Z}^S(x)$.

In the case of rigid boundary reflection:

$$\underline{Z}^S(x) = \frac{\underline{P}(x)}{\underline{V}_x(x)} = \frac{2\underline{P}^+\cos(kx)}{-j\dfrac{2\underline{P}^+}{z_0}\sin(kx)} = jz_0\cot(kx) \tag{2.26}$$

At a position $\lambda/4$ away from the reflector, i.e. $x = -\lambda/4$, $-3\lambda/4$, $-5\lambda/4$, $-7\lambda/4$ … ; $\underline{Z}^S=0$.

When $x = 0, -\lambda/2, -\lambda, -3\lambda/2 \ldots ; \underline{Z}^S = \infty$.