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Reading:

- For nonclassical light generation from parametric downconversion:
  - L. Mandel and E. Wolf *Optical Coherence and Quantum Optics,* (Cambridge University Press, Cambridge, 1995) sections 21.7, 22.4.
  - F.N.C. Wong, J.H. Shapiro, and T. Kim, "Efficient generation of polarization-entangled photons in a nonlinear crystal," Laser Phys. **16,** 1516 (2006).
- For Gaussian-state theory of parametric amplifier noise and its quantum signatures:
  - J.H. Shapiro and K.-X. Sun, "Semiclassical versus quantum behavior in fourth-order interference," J. Opt. Soc. Am. B **11**, 1130 (1994).
  - J.H. Shapiro, "Quantum Gaussian noise," Proc. SPIE **5111**, 382 (2003).

Introduction

In today's lecture we will continue and complete our analysis of spontaneous parametric downconversion (SPDC) by converting the classical treatment from Lecture 20 into a continuous-time field operator theory. As was done in Lecture 20, we shall assume continuous-wave (cw) pumping with no pump depletion, and a collinear type-II configuration in which the signal and idler fields are +z-going plane waves that are orthogonally polarized. Moreover, we shall assume that the signal and idler center frequencies are both $\omega_P/2$, i.e., half the pump frequency.[1] This frequency degeneracy of the signal and idler is not required for some nonclassical effects that can be obtained from SPDC, but is necessary for others, e.g., quadrature-noise squeezing. Thus it is worthwhile imposing this condition at the outset. Once we have established the

[1] Whereas the analysis in Lecture 20 assumed single-frequency signal and idler beams, the quantum theory requires that we include *all* frequencies, hence our identification of center frequencies for these beams.

quantum theory for SPDC, we will add cavity enhancement to convert the downconverter into an optical parametric amplifier (OPA). The OPA analysis that we shall perform will employ a simpler, lumped-element theory for the nonlinear interaction in the $\chi^{(2)}$ material that will quickly lead to a Gaussian-state characterization which gives rise to quadrature-noise squeezing. In Lecture 22, we shall finish our survey of the nonclassical signatures produced by $\chi^{(2)}$ interactions. There we shall consider Hong-Ou-Mandel interferometry and the generation of polarization-entangled photon pairs from SPDC, along with the photon-twins behavior of the signal and idler beams from an OPA.

Classical Theory of Spontaneous Parametric Downconversion

Slide 3 reprises our conceptual picture of spontaneous parametric downconversion. A strong, linearly-polarized (along $\vec{i}_P$) cw laser-beam pump at frequency ω_P is applied to the entrance facet (at $z = 0$) of a length-l crystalline material that possesses a $\chi^{(2)}$ nonlinearity. The action of the pump beam in conjunction with the crystal's nonlinearity couples lower-frequency signal and idler beams that we shall assume to be linearly polarized along orthogonal directions $\vec{i}_S = \vec{i}_x$ (signal) and $\vec{i}_I = \vec{i}_y$ (idler), respectively, with common center frequency $\omega_P/2$. In Lecture 20 we treated the signal, idler, and (non-depeleting) pump inside the crystal as monochromatic plane waves, with positive-frequency, photon-units fields given by

$$E_S^{(+)}(z,t) = A_S(z)e^{-j(\omega_P t/2 - k_S z)} \tag{1}$$

$$E_I^{(+)}(z,t) = A_I(z)e^{-j(\omega_P t/2 - k_I z)} \tag{2}$$

$$E_P^{(+)}(z,t) = A_P e^{-j(\omega_P t - k_P z)}. \tag{3}$$

respectively, for the polarization components of interest. In this representation, $\hbar\omega_P|A_S(z)|^2/2$ and $\hbar\omega_P|A_I(z)|^2/2$ are the signal and idler powers flowing across the z plane, for $0 \le z \le l$. For $z > l$, free-space propagation applies, i.e., the positive-frequency, photon-units signal, idler, pump fields in that region are

$$E_S^{(+)}(z,t) = A_S(l)e^{-j(\omega_P(t-(z-l)/c)/2 - k_S l)} \tag{4}$$

$$E_I^{(+)}(z,t) = A_I(l)e^{-j(\omega_P(t-(z-l)/c)/2 - k_I l)} \tag{5}$$

$$E_P^{(+)}(z,t) = A_P e^{-j(\omega_P(t-(z-l)/c) - k_P l)}. \tag{6}$$

The coupled-mode equations that the signal and idler satisfy inside the nonlinear crystal were shown last time to be

$$\frac{\partial A_S(z)}{\partial z} = j\kappa A_I^*(z)e^{j\Delta k z} \tag{7}$$

$$\frac{\partial A_I(z)}{\partial z} = j\kappa A_S^*(z)e^{j\Delta k z}, \tag{8}$$

for $0 \le z \le l$. Here: $\Delta k \equiv k_P(\omega_P) - k_S(\omega_P/2) - k_I(\omega_P/2)$ quantifies the phase-mismatch between the signal, idler, and pump beams in terms of their respective dispersion relations, $\{\, k_j(\omega) \equiv \omega n_j(\omega)/c : j = S, I, P\,\}$ with $\{\, n_j(\omega) : j = S, I, P\,\}$ denoting the refractive indices for the relevant polarization components; and

$$\kappa \equiv \sqrt{\frac{\hbar\omega_S\omega_I\omega_P}{2c^3\epsilon_0 n_S(\omega_S)n_I(\omega_I)n_P(\omega_P)\mathcal{A}}}\,\chi^{(2)}A_P \tag{9}$$

is a complex-valued coupling constant that is proportional to the pump's complex envelope and the crystal's second-order nonlinear susceptibility. The general solution to these equations is

$$A_S(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_S(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_I^*(0)\right]e^{j\Delta kl/2} \tag{10}$$

$$A_I(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_I(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_S^*(0)\right]e^{j\Delta kl/2}, \tag{11}$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\Delta k/2)^2}. \tag{12}$$

However, to get the most efficient interaction, we need phase-matched operation, i.e., $\Delta k = 0$, in which case the solution to Eqs. (7) and (8) reduces to

$$A_S(l) = \cosh(|\kappa|l)A_S(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_I^*(0) \tag{13}$$

$$A_I(l) = \cosh(|\kappa|l)A_I(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_S^*(0), \tag{14}$$

indicating increasing amounts of signal-idler coupling with increasing $|\kappa|l$, i.e., with increasing pump power or crystal length.

Quantum Theory of Spontaneous Parametric Downconversion

At the end of Lecture 20 we noted that the SPDC's frequency-sum condition, $\omega_P = \omega_S + \omega_I$, and its phase-matching condition, $k_P = k_S + k_I$, could be interpreted as energy conservation and momentum conservation, respectively, for a photon fission process in which a single pump photon divides into a signal photon and an idler photon. We also noted, in that Lecture, that the solutions to the coupled-mode equations, which we reprised in the previous section, are a two-mode Bogoluibov transformation, similar to what we saw earlier in the semester for our two-mode optical parametric amplifier. It is now time for us to go beyond these precursors and establish the quantum field-operator theory for cw collinear SPDC at frequency degeneracy.[2]

[2]The basic concepts we shall develop can be extended to non-degenerate, non-collinear operation, but we shall not do so.

Suppose that $\hat{E}_S^{(+)}(z,t)$ and $\hat{E}_I^{(+)}(z,t)$ for $0 \le z \le l$ are the positive-frequency, photon-units $+z$-going plane-wave field operators for the $\vec{i}_x$ and $\vec{i}_y$ polarization components of the signal and idler, respectively.[3] Because we must preserve δ-function commutators for the signal and idler field operators leaving the nonlinear crystal, we must include all frequencies in them. Hence we shall take $\hat{E}_S^{(+)}(z,t)$ and $\hat{E}_I^{(+)}(z,t)$ to have the following Fourier decompositions:

$$\hat{E}_S^{(+)}(z,t) = \int \frac{\mathrm{d}\omega}{2\pi}\, \hat{A}_S(z,\omega) e^{-j[(\omega_P/2+\omega)t - k_S(\omega_P/2+\omega)z]}, \tag{15}$$

$$\hat{E}_I^{(+)}(z,t) = \int \frac{\mathrm{d}\omega}{2\pi}\, \hat{A}_I(z,\omega) e^{-j[(\omega_P/2-\omega)t - k_I(\omega_P/2-\omega)z]}. \tag{16}$$

In these expressions, $\hat{A}_S(z,\omega)$ is the plane-wave field-component annihilation operator for the signal beam at frequency shift ω from frequency degeneracy, and $\hat{A}_I(z,\omega)$ is the plane-wave field-component annihilation operator for the idler beam at frequency shift $-\omega$ from frequency degeneracy.[4] At the crystal's entrance and exit facets, the signal and idler fields operators must have the following non-zero commutators that apply for free-space fields,

$$[\hat{E}_S^{(+)}(z,t), \hat{E}_S^{(+)\dagger}(z,u)] = [\hat{E}_I^{(+)}(z,t), \hat{E}_I^{(+)\dagger}(z,u)] = \delta(t-u), \quad \text{for } z = 0, l, \tag{17}$$

which imply that

$$[\hat{A}_S(z,\omega), \hat{A}_S^\dagger(z,\omega')] = [\hat{A}_I(z,\omega), \hat{A}_I^\dagger(z,\omega')] = 2\pi\delta(\omega-\omega'), \quad \text{for } z = 0, l, \tag{18}$$

are the only non-zero frequency-domain commutators at the crystal's input and output. Any proper quantized form of the coupled-mode equations and their solutions must preserve these commutator brackets.

We shall assume that the downconverter is phase-matched at frequency degeneracy, viz.,

$$\Delta k(\omega) \equiv k_P(\omega_P) - k_S(\omega_P/2+\omega) - k_I(\omega_P/2-\omega), \tag{19}$$

satisifes $\Delta k(0) = 0$, and that group-velocity dispersion can be neglected, so that

$$\Delta k(\omega) \approx \omega \Delta k' \tag{20}$$

[3] A full field-operator treatment should include *all* spatial modes, not just the $+z$-going plane-wave modes, and *both* polarizations for all such modes. However, we shall limit our consideration to these polarizations of the $+z$-going signal and idler plane waves. For coherent (homodyne or heterodyne) detection measurements, spatial and polarization mode selection automatically occurs by choice of the local oscillator, so our assumption is easily enforced in such measurement scenarios. For direct detection, however, other spatial modes and polarizations may have to be included, depending on the SPDC and measurement configuration.

[4] This sign convention is convenient because the coupled-mode equations for classical versions of these Fourier decompositions link $A_S(z,\omega)$ to $A_I^*(z,\omega)$ and vice versa.

holds, where

$$\Delta k' \equiv \left.\frac{\mathrm{d}\Delta k}{\mathrm{d}\omega}\right|_{\omega=0} = -\left.\frac{\mathrm{d}k_S(\omega_P/2+\omega)}{\mathrm{d}\omega}\right|_{\omega=0} - \left.\frac{\mathrm{d}k_I(\omega_P/2-\omega)}{\mathrm{d}\omega}\right|_{\omega=0}. \tag{21}$$

Emboldened by last lecture's comment about Bogoliubov transformations, as well as our earlier quantization of the classical harmonic oscillator, we shall *assume* that $\hat{A}_S(z,\omega)$ and $\hat{A}_I(z,\omega)$ obey the following coupled-mode equations:

$$\frac{\partial \hat{A}_S(z,\omega)}{\partial z} = j\kappa \hat{A}_I^\dagger(z,\omega)e^{j\omega\Delta k' z} \tag{22}$$

$$\frac{\partial \hat{A}_I(z,\omega)}{\partial z} = j\kappa \hat{A}_S^\dagger(z,\omega)e^{j\omega\Delta k' z}, \tag{23}$$

for $0 \le z \le l$, where κ is the *same* coupling constant from the classical theory, i.e., Eq. (9).[5] These equations have the following solution, cf. Eqs. (10) and (11):

$$\hat{A}_S(l,\omega) = \left[\left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)\hat{A}_S(0,\omega) + j\kappa l\frac{\sinh(pl)}{pl}\,\hat{A}_I^\dagger(0,\omega)\right]e^{j\omega\Delta k' l/2} \tag{24}$$

$$\hat{A}_I(l,\omega) = \left[\left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)\hat{A}_I(0,\omega) + j\kappa l\frac{\sinh(pl)}{pl}\,\hat{A}_S^\dagger(0,\omega)\right]e^{j\omega\Delta k' l/2}, \tag{25}$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\omega\Delta k'/2)^2}. \tag{26}$$

To verify that these solution preserve free-space commutator brackets, let us define

$$\mu(\omega) = \left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)e^{j\omega\Delta k' l/2} \tag{27}$$

$$\nu(\omega) = j\kappa l\frac{\sinh(pl)}{pl}e^{j\omega\Delta k' l/2}, \tag{28}$$

so that Eqs. (24) and (25) become

$$\hat{A}_S(l,\omega) = \mu(\omega)\hat{A}_S(0,\omega) + \nu(\omega)\hat{A}_I^\dagger(0,\omega) \tag{29}$$

$$\hat{A}_I(l,\omega) = \mu(\omega)\hat{A}_I(0,\omega) + \nu(\omega)\hat{A}_S^\dagger(0,\omega). \tag{30}$$

[5] We have assumed that the strong, non-depleting pump is in a coherent state such that—as in the case of the local oscillator beam for homodyne and heterodyne detection—it acts classically in SPDC.

Now, because

$$|\mu(\omega)|^2 - |\nu(\omega)|^2 = \left[\cosh^2(pl) + \left(\frac{\omega\Delta k'}{2p}\right)^2 \sinh^2(pl)\right] - \left(\frac{\kappa}{p}\right)^2 \sinh^2(pl) \quad (31)$$

$$= \cosh^2(pl) - \sinh^2(pl) = 1, \quad (32)$$

Eqs. (29) and (30) are a two-mode Bogoliubov transformation that ensures proper commutator preservation.[6]

Gaussian-State Characterization of SPDC

Equations (29) and (30) allow us an immediate insight into the joint state of the signal and idler produced by spontaneous parametric downconversion, i.e., the joint state of the signal and idler beams emerging from the crystal at $z = l$ when the signal and idler inputs at $z = 0$ are in their vacuum states. In particular, the linearity of these equations, combined with the fact that the vacuum state is zero-mean and Gaussian, tells us that the signal and idler outputs will be in a zero-mean jointly Gaussian state. Hence they are completely characterized by their phase-insensitive and phase-sensitive correlation functions, of which the only non-zero ones are $\langle \hat{A}_S^\dagger(l,\omega)\hat{A}_S(l,\omega')\rangle$, $\langle \hat{A}_I^\dagger(l,\omega)\hat{A}_I(l,\omega')\rangle$, and $\langle \hat{A}_S(l,\omega)\hat{A}_I(l,\omega')\rangle$. These correlations are easily computed, e.g., for the signal's phase-insensitive correlation function we have that

$$\begin{aligned}
&\langle \hat{A}_S^\dagger(l,\omega)\hat{A}_S(l,\omega')\rangle \\
&= \langle [\mu^*(\omega)\hat{A}_S^\dagger(0,\omega) + \nu^*(\omega)\hat{A}_I(0,\omega)][\mu(\omega')\hat{A}_S(0,\omega') + \nu(\omega')\hat{A}_I^\dagger(0,\omega')]\rangle \quad (33) \\
&= \mu^*(\omega)\mu(\omega')\langle \hat{A}_S^\dagger(0,\omega)\hat{A}_S(0,\omega')\rangle + \mu^*(\omega)\nu(\omega')\langle \hat{A}_S^\dagger(0,\omega)\hat{A}_I^\dagger(0,\omega')\rangle \\
&+ \nu^*(\omega)\mu(\omega')\langle \hat{A}_I(0,\omega)\hat{A}_S(0,\omega')\rangle + \nu^*(\omega)\nu(\omega')\langle \hat{A}_I(0,\omega)\hat{A}_I^\dagger(0,\omega')\rangle. \quad (34)
\end{aligned}$$

Now, because the input fields are in their vacuum states, all their normally-ordered correlation functions vanish, so, using the commutator (18), we get

$$\langle \hat{A}_I(0,\omega)\hat{A}_I^\dagger(0,\omega')\rangle = 2\pi\delta(\omega-\omega'), \quad (35)$$

whence

$$\langle \hat{A}_S^\dagger(l,\omega)\hat{A}_S(l,\omega')\rangle = 2\pi|\nu(\omega)|^2\delta(\omega-\omega'). \quad (36)$$

Similar calculations yield

$$\langle \hat{A}_I^\dagger(l,\omega)\hat{A}_I(l,\omega')\rangle = 2\pi|\nu(\omega)|^2\delta(\omega-\omega'), \quad (37)$$

[6]Our proof has assumed that p is real valued, i.e., it applies for frequencies low enough to give $|\omega\Delta k'/2| \le |\kappa|$. At higher frquencies, where $|\omega\Delta k'/2| > |\kappa|$ prevails, p becomes imaginary, but a similar calculation—left to the reader—will show that Eqs. (29) and (30) still constitute a two-mode Bogoliubov transformation and hence commutator preserving.

and

$$\langle \hat{A}_S(l,\omega)\hat{A}_I(l,\omega')\rangle = 2\pi\mu(\omega)\nu(\omega)\delta(\omega-\omega'), \tag{38}$$

for the other correlation functions that we need.

For future use it will be valuable to find the phase-insensitive and phase-sensitive correlation functions for the baseband signal and idler field operators defined by

$$\hat{E}_S^{(+)}(l,t) = \hat{E}_S(t)e^{-j(\omega_P t/2 - k_S(\omega_P/2)l)} \quad \text{and} \quad \hat{E}_I^{(+)}(l,t) = \hat{E}_I(t)e^{-j(\omega_P t/2 - k_I(\omega_P/2)l)}. \tag{39}$$

Using the Fourier relations

$$\hat{E}_S(t) = \int \frac{\mathrm{d}\omega}{2\pi}\,\hat{A}_S(l,\omega)e^{-j\omega(t-k'_S l)}, \tag{40}$$

$$\hat{E}_I(t) = \int \frac{\mathrm{d}\omega}{2\pi}\,\hat{A}_I(l,\omega)e^{j\omega(t+k'_I l)}, \tag{41}$$

where

$$k'_S \equiv \left.\frac{\mathrm{d}k_S(\omega_P/2+\omega)}{\mathrm{d}\omega}\right|_{\omega=0} \quad \text{and} \quad k'_I \equiv \left.\frac{\mathrm{d}k_I(\omega_P/2-\omega)}{\mathrm{d}\omega}\right|_{\omega=0}, \tag{42}$$

together with the frequency-domain correlation functions derived above, we find that the non-zero correlations of the baseband field operators are stationary—dependent on time-difference only—and given by

$$K_{SS}^{(n)}(\tau) \equiv \langle \hat{E}_S^\dagger(t+\tau)\hat{E}_S(t)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,|\nu(\omega)|^2 e^{j\omega\tau} \tag{43}$$

$$K_{II}^{(n)}(\tau) \equiv \langle \hat{E}_I^\dagger(t+\tau)\hat{E}_I(t)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,|\nu(-\omega)|^2 e^{j\omega\tau} \tag{44}$$

$$K_{SI}^{(p)}(\tau) \equiv \langle \hat{E}_S(t+\tau)\hat{E}_I(t)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,\mu(-\omega)\nu(-\omega)e^{j\omega(\tau+\Delta k' l)}, \tag{45}$$

with $^{(n)}$ denoting the phase-insensitive (normally-ordered) auto-correlation functions and $^{(p)}$ denoting the phase-sensitive cross-correlation function. We have made all of these expressions employ $e^{j\omega\tau}$ inverse Fourier kernels so that—in keeping with our definition of noise spectral densities for real-valued classical random processes—we can say that

$$\mathcal{S}_{SS}^{(n)}(\omega) = |\nu(\omega)|^2,\ \mathcal{S}_{II}^{(n)}(\omega) = |\nu(-\omega)|^2,\ \text{and } \mathcal{S}_{SI}^{(p)}(\omega) = \mu(-\omega)\nu(-\omega)e^{-j\omega\Delta k' l}, \tag{46}$$

are their corresponding spectral densities.

Physically, $\mathcal{S}_{SS}^{(n)}(\omega)/2\pi$ is the average photon-flux per unit bilateral bandwidth (in rad/s) in the signal beam at frequency $\omega_P/2+\omega$, and $\mathcal{S}_{II}^{(n)}(\omega)/2\pi$ is the average photon-flux per unit bilateral bandwidth (in rad/s) in the idler beam at frequency $\omega_P/2-\omega$. These functions are usually referred to as the fluorescence spectra of the signal and idler, respectively. SPDC is usually performed in the regime wherein $|\kappa|l \ll 1$ so that we can employ $p \approx j\omega|\Delta k'|/2$ at all relevant detunings from degeneracy, i.e., for all ω values of interest. As noted in Lecture 20, this low-gain condition leads to the following approximations for the Bogoliubov functions,[7]

$$\mu(\omega) \approx 1 \quad \text{and} \quad \nu(\omega) \approx j\kappa l \frac{\sin(\omega\Delta k' l/2)}{\omega\Delta k' l/2} e^{j\omega\Delta k' l/2}. \tag{47}$$

It follows that the signal and idler fluorescence spectra are equal, and given by

$$\mathcal{S}^{(n)}_{SS}(\omega) = \mathcal{S}^{(n)}_{II}(\omega) \approx |\kappa|^2 l^2 \left(\frac{\sin(\omega\Delta k' l/2)}{\omega\Delta k' l/2}\right)^2. \tag{48}$$

Thus, they peak at $\omega = 0$, i.e., frequency degeneracy, where the phase-matching condition is satisfied. More importantly, we see that these fluorescence spectra are consistent with the photon fission interpretation of SPDC, in that the signal beam's fluorescence spectrum at $\omega_P/2 + \omega$ equals the idler beam's fluorescence spectrum at $\omega_P/2 - \omega$. The phase-sensitive cross-spectral density, $\mathcal{S}^{(p)}_{SI}(\omega)$, in the low-gain regime, is

$$\mathcal{S}^{(p)}_{SI}(\omega) \approx j\kappa l \frac{\sin(\omega\Delta k' l/2)}{\omega\Delta k' l/2} e^{j\omega\Delta k' l/2}. \tag{49}$$

We shall work further with these low-gain spectra, and their associated correlation functions, in Lecture 22, when we study the Hong-Ou-Mandel dip and SPDC generation of polarization-entangled photon pairs. For the rest of today's lecture, however, we will turn our attention to cavity-enhanced SPDC, i.e., the optical parametric amplifier.

The Doubly-Resonant Optical Parametric Amplifier

To go beyond the low-gain regime in cw SPDC we need the optical parametric amplifier (OPA), shown schematically on slide 10 as a $\chi^{(2)}$ crystal inside an optical cavity formed by two mirrors. These mirrors are anti-reflection coated for the pump frequency ω_P, so the pump makes a single pass, from left to right, through through the crystal. We will assume that the mirror on the left is a perfect reflector at the frequency $\omega_P/2$, while the mirror on the right is lossless and highly reflecting at this frequency. As a result, the spontaneously generated signal and idler photons—resulting from frequency-degenerate downconversion in the $\chi^{(2)}$ crystalb— ounce back and forth between the mirrors many times before exiting through the highly-reflecting mirror. This optical feedback process greatly enhances the nonlinear interaction by making the crystal act as though it was much longer than it is. Of course, this feedback is only effective when it is *positive* feedback, which in this case means that $\omega_P/2$

[7]These approximations violate strict commutator preservation, i.e., $|\mu(\omega)|^2 - |\nu(\omega)|^2 = 1$ is only satisfied to first order in $|\kappa|$.

must be a resonant frequency of the cavity, i.e., the roundtrip phase delay inside the cavity at frequency $\omega_P/2$ must be an integer multiple of 2π. In what follows we shall assume that the cavity is resonant for both the signal and idler polarizations at frequency $\omega_P/2$.

Although it is possible to analyze this OPA arrangement by imposing cavity mirrors around the SPDC analysis we've given earlier in this lecture, a much simpler route to getting to the essential physics employs a lumped-element treatment for intracavity modes that are resonant at frequency $\omega_P/2$ for both the signal and idler ($\vec{i}_x$ and $\vec{i}_y$) polarizations. We shall use $\hat{E}_S^{\rm in}(t)$ and $\hat{E}_I^{\rm in}(t)$ to denote the vacuum-state, baseband field operators of the relevant signal and idler polarizations that are incident on the cavity in slide 10 from the right, while $\hat{a}_S(t)$ and $\hat{a}_I(t)$ will be the photon annihilation operators for the associated intracavity modes.[8] The equations of motion for the OPA system then turn out to be

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\Gamma\right)\hat{a}_S(t) = G\Gamma\hat{a}_I^\dagger(t)+\sqrt{2\Gamma}\hat{E}_S^{\rm in}(t) \tag{50}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\Gamma\right)\hat{a}_I(t) = G\Gamma\hat{a}_S^\dagger(t)+\sqrt{2\Gamma}\hat{E}_I^{\rm in}(t), \tag{51}$$

where $0 < G < 1$ is the normalized OPA gain[9] and $\Gamma > 0$ is the linewidth of the signal and idler intracavity modes. Once Eqs. (50) and (51) have been solved for the intracavity modes as functions of the input field operators, the baseband field operators for the signal and idler outputs follow from

$$\hat{E}_S^{\rm out}(t) = \sqrt{2\Gamma}\hat{a}_S(t) - \hat{E}_S^{\rm in}(t) \quad \text{and} \quad \hat{E}_I^{\rm out}(t) = \sqrt{2\Gamma}\hat{a}_I(t) - \hat{E}_I^{\rm in}(t). \tag{52}$$

Frequency-domain techniques~~as~~ we used above to obtain our SPDC input-output relations~~can~~ be used to derive the following two-mode Bogoliubov relation between the Fourier transforms[10] of the input and output field operators,

$$\begin{aligned}
\hat{\mathcal{E}}_S^{\rm out}(\Omega) &= \mu(\Omega)\hat{\mathcal{E}}_S^{\rm in}(\Omega)+\nu(\Omega)\hat{\mathcal{E}}_I^{\rm in\dagger}(\Omega) && (53)\\
\hat{\mathcal{E}}_I^{\rm out}(\Omega) &= \mu^*(\Omega)\hat{\mathcal{E}}_I^{\rm in}(\Omega)+\nu^*(\Omega)\hat{\mathcal{E}}_S^{\rm in\dagger}(\Omega), && (54)
\end{aligned}$$

where

$$\begin{aligned}
\mu(\Omega) &\equiv \frac{1+G^2+\Omega^2/\Gamma^2}{1-G^2-\Omega^2/\Gamma^2-2j\Omega/\Gamma} && (55)\\
\nu(\Omega) &\equiv \frac{2G}{1-G^2-\Omega^2/\Gamma^2-2j\Omega/\Gamma}. && (56)
\end{aligned}$$

[8]The field operators $\hat{E}_m^{\rm in}(t)$ for $m = S, I$ have the usual δ-function commutator with their adjoints, $[\hat{E}_m^{\rm in}(t), \hat{E}_m^{\rm in\dagger}(u)] = \delta(t-u)$ for $m = S, I$, while the intracavity annihilation operators $\hat{a}_m(t)$ for $m = S, I$ have the canonical commutation relation, $[\hat{a}_m(t), \hat{a}_m^\dagger(t)] = 1$ for $m = S, I$, with their adjoints.

[9]Here, $G^2 = P_P/P_T$, where P_P is the pump power and P_T is the threshold power, i.e., the pump power value for which the OPA breaks into oscillation and becomes an optical parametric oscillator.

[10]Our sign convention for these transforms is $\hat{\mathcal{E}}_S(\Omega) = \int\mathrm{d}t\,\hat{E}_S(t)e^{j\Omega t}$ and $\hat{\mathcal{E}}_I(\Omega) = \int\mathrm{d}t\,\hat{E}_I(t)e^{-j\Omega t}$.

It easily shown that $|\mu(\Omega)|^2 - |\nu(\Omega)|^2 = 1$ and that Eqs. (53) and (54) give rise to the proper commutator brackets. More importantly, Eqs.(53) and (54) are linear and their driving terms are vacuum-state field operators. It follows that $\hat{E}_S^{\rm out}(t)$ and $\hat{E}_I^{\rm out}(t)$ will be in a zero-mean jointly Gaussian state. Paralleling the approach used to find the correlation functions for spontaneous parametric downconversion, we can show that this jointly Gaussian state is completely characterized by the following spectral densities and stationary correlation functions:

$$\begin{aligned} \mathcal{S}_{mm}^{(n)}(\Omega) &= \int d\tau\, K_{mm}^{(n)}(\tau)e^{-j\Omega\tau} = |\nu(\Omega)|^2 && (57)\\ &= \frac{4G^2}{(1-G^2-\Omega^2/\Gamma^2)^2 + 4\Omega^2/\Gamma^2}, \quad \text{for } m = S, I, && (58)\\ \mathcal{S}_{SI}^{(p)}(\Omega) &= \int d\tau\, K_{SI}^{(p)}(\tau)e^{-j\Omega\tau} = \mu^*(\Omega)\nu(\Omega) && (59)\\ &= \frac{2G(1+G^2+\Omega^2/\Gamma^2)}{(1-G^2-\Omega^2/\Gamma^2)^2 + 4\Omega^2/\Gamma^2}, && (60)\end{aligned}$$

and

$$K_{mm}^{(n)}(\tau) = \langle \hat{E}_m^{\rm out\dagger}(t)\hat{E}_m^{\rm out}(u)\rangle = \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} - \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right], \text{ for } m = S, I, \quad (61)$$

$$K_{SI}^{(p)}(\tau) = \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} + \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right]. \quad (62)$$

In the next section, we will show how the preceding spectra lead to quadrature-noise squeezing.

Quadrature-Noise Squeezing from an OPA

From our previous work on two-mode parametric amplifiers, we expect that the $\pm 45^\circ$ polarizations at the output of our continuous-time OPA should exhibit quadrature-noise squeezing. Let's show that this is so for the $+45^\circ$ case. The baseband field operator for this polarization is

$$\hat{E}_{+45}^{\rm out}(t) \equiv \frac{\hat{E}_S^{\rm out}(t) + \hat{E}_I^{\rm out}(t)}{\sqrt{2}}. \quad (63)$$

This field operator is in a zero-mean Gaussian state whose phase-insensitive and phase-sensitive correlation functions are

$$\begin{aligned} K^{(n)}(\tau) &\equiv \langle \hat{E}_{+45}^{\rm out\dagger}(t+\tau)\hat{E}_{+45}^{\rm out}(t)\rangle = \frac{K_{SS}^{(n)}(\tau) + K_{II}^{(n)}(\tau)}{2} && (64)\\ &= \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} - \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right], && (65)\end{aligned}$$

and

$$K^{(p)}(\tau) \equiv \langle \hat{E}^{\rm out}_{+45}(t+\tau)\hat{E}^{\rm out}_{+45}(t)\rangle = \frac{K^{(p)}_{SI}(\tau) + K^{(p)}_{II}(-\tau)}{2} \quad (66)$$

$$= \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} + \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right], \quad (67)$$

respectively. The spectral densities associated with these correlation functions are

$$\mathcal{S}^{(n)}(\Omega) \equiv \int d\tau\, K^{(n)}(\tau)e^{-j\Omega\tau} = |\nu(\Omega)|^2 \quad (68)$$

$$\mathcal{S}^{(p)}(\Omega) \equiv \int d\tau\, K^{(p)}(\tau)e^{-j\Omega\tau} = \mu^*(\Omega)\nu(\Omega). \quad (69)$$

Now, consider the balanced homodyne measurement systemsho wn on slide 11– for detecting the θ-quadrature of $\hat{E}^{\rm out}_{+45}(t)$. Here we have assumed unity quantum efficiency photodetectors, and omitted the low-pass filter. From our continuous-time theory of homodyne detection we know that the photocurrent difference $\Delta i(t)$ has statistics that are equivalent to those of the operator

$$\Delta\hat{i}(t) = 2q\sqrt{\frac{P_{\rm LO}}{\hbar\omega_P/2}}\,{\rm Re}[\hat{E}^{\rm out}_{+45}(t)e^{-j\theta}]. \quad (70)$$

Because $\hat{E}^{\rm out}_{+45}(t)$ is in a zero-mean, statistically-stationary Gaussian state, the homodyne measurement will yield a zero-mean, stationary Gaussian random process whose covariance function is

$$K_{\Delta i\Delta i}(\tau) \equiv \langle \Delta i(t+\tau)\Delta i(t)\rangle \quad (71)$$

$$= q^2\frac{P_{\rm LO}}{\hbar\omega_P/2}\{1 + K^{(n)}(\tau) + K^{(n)}(-\tau) + 2{\rm Re}[K^{(p)}(\tau)e^{-2j\theta}]\}. \quad (72)$$

The photocurrent-noise spectral density that will be observed using a spectrum analyzer at the homodyne system's output is thus

$$\mathcal{S}_{\Delta i\Delta i}(\Omega) = \int d\tau\, K_{\Delta i\Delta i}(\tau)e^{-j\Omega\tau} \quad (73)$$

$$= q^2\frac{P_{\rm LO}}{\hbar\omega_P/2}[1 + 2|\nu(\Omega)|^2 + 2{\rm Re}(\mu^*(\Omega)\nu(\Omega)e^{-2j\theta})] \quad (74)$$

$$= q^2\frac{P_{\rm LO}}{\hbar\omega_P/2}|\mu(\Omega) + \nu(\Omega)e^{-2j\theta}|^2. \quad (75)$$

Were $\hat{E}^{\rm out}_{+45}(t)$ in a coherent state, this homodyne receiver's photocurrent-noise spectral density would be

$$\mathcal{S}_{\Delta i\Delta i}(\Omega)|_{\rm CS} = q^2\frac{P_{\rm LO}}{\hbar\omega_P/2}, \quad (76)$$

representing the shot-noise limit of semiclassical theory. The normalized photocurrent-noise spectral density,

$$\frac{\mathcal{S}_{\Delta i \Delta i}(\Omega)}{\mathcal{S}_{\Delta i \Delta i}(\Omega)|_{\text{CS}}} = |\mu(\Omega) + \nu(\Omega)e^{-2j\theta}|^2, \tag{77}$$

contains contains phase-sensitive noise that, as shown in the left panel on slide 12, goes well below the shot-noise level at $\theta = \pm\pi/2$ for $\Omega = 0$. As shown in the right panel on slide 12, the strongest quadrature-noise squeezing is limited to frequencies below the cavity linewidth.

The Road Ahead

In the next lecture we shall use the results developed today for SPDC and the OPA to study additional signatures of nonclassical light that can be obtained from these nonlinear optical systems. Of particular interest will be Hong-Ou-Mandel interferometry, as it relates to the important notion of distinguishability. We will also connect our treatment of SPDC with the concept of a biphoton.