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6.453 QUANTUM OPTICAL COMMUNICATION

**Problem Set 3**

Fall 2008

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**Problem 3.1**

Here we shall extend the results of problem 2.2 to include classically-random polarizations. Suppose we have a  $+z$ -propagating, frequency- $\omega$  photon whose polarization vector (in problem 2.1 notation) is,

$$\mathbf{i} \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

where  $\alpha_x$  and  $\alpha_y$  are a pair of complex-valued classical random variables that satisfy

$$|\alpha_x|^2 + |\alpha_y|^2 = 1,$$

with probability one. (Two joint complex-valued random variables,  $\alpha_x$  and  $\alpha_y$ , are really four joint real-valued random variables, viz., the real and imaginary parts of  $\alpha_x$  and  $\alpha_y$ .)

The Poincaré sphere representation for the *average* behavior of this random polarization vector is,

$$\mathbf{r} \equiv \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2\text{Re}[\langle \alpha_x^* \alpha_y \rangle] \\ 2\text{Im}[\langle \alpha_x^* \alpha_y \rangle] \\ \langle |\alpha_x|^2 \rangle - \langle |\alpha_y|^2 \rangle \end{bmatrix},$$

where—in keeping with the quantum notation for averages— $\langle \cdot \rangle$  denotes ensemble average.

- (a) Use the Schwarz inequality to prove that  $\mathbf{r}^T \mathbf{r} \equiv r_1^2 + r_2^2 + r_3^2 \leq 1$ , i.e., the  $\mathbf{r}$  vector lies on or inside the unit sphere.
- (b) Let  $\mathbf{i}_a$  and  $\mathbf{i}_b$  be a pair of deterministic, orthogonal, complex-valued unit vectors, viz.,

$$\mathbf{i}_k^\dagger \mathbf{i}_l = \delta_{kl} \equiv \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases},$$

where  $k$  and  $l$  can each be either  $a$  or  $b$ . By means of wave plates, a polarizing beam splitter, and a pair of ideal photon counters, it is possible to measure whether the photon is polarized along  $\mathbf{i}_a$  or along  $\mathbf{i}_b$ . The statistics of this measurement satisfy,

$$\text{Pr}(\text{polarized along } \mathbf{i}_a) = \frac{1 + \mathbf{r}_a^T \mathbf{r}}{2}, \tag{1}$$

$$\text{Pr}(\text{polarized along } \mathbf{i}_b) = \frac{1 + \mathbf{r}_b^T \mathbf{r}}{2}, \tag{2}$$

where  $\mathbf{r}_a$  and  $\mathbf{r}_b$  are the Poincaré sphere representations of  $\mathbf{i}_a$  and  $\mathbf{i}_b$ , respectively. Show that  $\mathbf{r}_a = -\mathbf{r}_b$ , so that Eqs. (1) and (2) constitute a proper probability distribution.

- (c) Suppose that the photon's random polarization leads to  $\mathbf{r} = \mathbf{0}$ , i.e.,  $r_1 = r_2 = r_3 = 0$ . Show that

$$\Pr(\text{polarized along } \mathbf{i}_a) = \Pr(\text{polarized along } \mathbf{i}_b) = \frac{1}{2},$$

for all pairs of deterministic, orthogonal complex-valued unit vectors  $\{\mathbf{i}_a, \mathbf{i}_b\}$ , and thus that  $\mathbf{r} = \mathbf{0}$  represents a state of completely random polarization. Contrast the preceding measurement statistics with what will be obtained when

$$\mathbf{r} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{r}_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{r}_b = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

and when

$$\mathbf{r} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{r}_a = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{r}_b = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$

are the Poincaré sphere representations of the photon and the pair of orthogonal polarizations being measured.

### Problem 3.2

Here we introduce the notion of a density operator, i.e., a way to account for classical randomness limiting our knowledge of a quantum system's state. Consider a quantum mechanical system whose state at time  $t$  is not known. Instead, there is a classically-random probability distribution for this state. In particular, suppose that there are  $M$  distinct unit-length kets,  $\{|\psi_m\rangle : 1 \leq m \leq M\}$ , and that the system is known to be in one of these states. Moreover the probability that it is in state  $|\psi_m\rangle$  at time  $t$  is  $p_m$ , for  $1 \leq m \leq M$ , where  $\{p_m : 1 \leq m \leq M\}$  is a classical probability distribution:  $p_m \geq 0$  and  $\sum_{m=1}^M p_m = 1$ .

- (a) Suppose that we measure the observable  $\hat{O}$  at time  $t$ , where  $\hat{O}$  has distinct eigenvalues,  $\{o_n : 1 \leq n < \infty\}$ , and a complete orthonormal set of associated eigenkets,  $\{|o_n\rangle : 1 \leq n < \infty\}$ . GIVEN that the state of the system at time  $t$  is  $|\psi_m\rangle$ , we know that the  $\hat{O}$  measurement will yield outcome  $o_n$  with conditional probability  $\Pr(o_n | |\psi_m\rangle) \equiv |\langle o_n | \psi_m \rangle|^2$ , for  $1 \leq n < \infty$  and  $1 \leq m \leq M$ . Use this conditional probability distribution to obtain the unconditional probability,  $\Pr(o_n)$ , of getting the outcome  $o_n$  when we make the  $\hat{O}$  measurement at time  $t$ .

(b) Define a density operator for the system by,

$$\hat{\rho} \equiv \sum_{m=1}^M p_m |\psi_m\rangle \langle \psi_m|.$$

Show that  $\hat{\rho}$  is an Hermitian operator, and verify that your answer to (a) can be reduced to

$$\Pr(o_n) = \langle o_n | \hat{\rho} | o_n \rangle, \quad \text{for } 1 \leq n < \infty.$$

(c) Show that the expected value of the  $\hat{O}$  measurement, i.e.,

$$\langle \hat{O} \rangle \equiv \sum_{n=1}^{\infty} o_n \Pr(o_n),$$

satisfies

$$\langle \hat{O} \rangle = \text{tr}(\hat{\rho} \hat{O}),$$

where  $\text{tr}(\hat{A})$  for any linear Hilbert-space operator,  $\hat{A}$ , is the trace of that operator, defined as follows. Let  $\{|k\rangle : 1 \leq k < \infty\}$  be an arbitrary complete set of orthonormal kets on the quantum system's state space, so that

$$\hat{I} = \sum_{k=1}^{\infty} |k\rangle \langle k|.$$

Then

$$\text{tr}(\hat{A}) \equiv \sum_{k=1}^{\infty} \langle k | \hat{A} | k \rangle,$$

i.e., it is the sum of the operator's diagonal matrix-elements in the  $\{|k\rangle\}$  representation. **Comment:** The trace operation is invariant to the choice of the CON basis used for its calculation. Hence a propitious choice of the basis can be a great aid in simplifying the calculation of averages involving a density operator.

### Problem 3.3

Here we will explore the difference between a pure state and a mixed state, i.e., the difference between knowing that a quantum system is in a definite state  $|\psi\rangle$  as opposed to having a classically-random distribution over a set of such states, namely a density operator  $\hat{\rho}$ . Because the density operator is Hermitian, it has eigenvalues and eigenkets. Let us assume that these form a countable set, viz.,  $\hat{\rho}$  has eigenvalues,  $\{\rho_n : 1 \leq n < \infty\}$ , and associated eigenkets  $\{|\rho_n\rangle : 1 \leq n < \infty\}$ , that satisfy

$$\hat{\rho} |\rho_n\rangle = \rho_n |\rho_n\rangle, \quad \text{for } 1 \leq n < \infty.$$

Without loss of generality, we can assume that these eigenkets form a complete orthonormal set, i.e.,

$$\langle \rho_m | \rho_n \rangle = \delta_{nm},$$

$$\hat{I} = \sum_{n=1}^{\infty} |\rho_n\rangle \langle \rho_n|.$$

(a) Show that the eigenvalues  $\{\rho_n\}$  satisfy

$$0 \leq \rho_n \leq 1, \quad \text{for } 1 \leq n < \infty,$$

and

$$\sum_{n=1}^{\infty} \rho_n = 1.$$

(b) Show that  $\text{tr}(\hat{\rho}) = 1$  for any density operator

(c) Suppose that the quantum system is in a pure state, i.e., it is known to be in the state  $|\psi\rangle$ . Show that this situation can be represented in density-operator form by setting  $\rho_1 = 1$  and  $|\rho_1\rangle = |\psi\rangle$ , viz., a pure state has a density operator with only one eigenket whose associated eigenvalue is non-zero. Show that  $\text{tr}(\hat{\rho}^2) = 1$  for any pure-state density operator.

(d) When the density operator has two or more eigenkets with non-zero eigenvalues we say that the state is mixed, i.e., there are at least two different pure states that can occur with non-zero probabilities. Show that  $\text{tr}(\hat{\rho}^2) < 1$  for any mixed-state density operator.

### Problem 3.4

Let  $\hat{A}$  and  $\hat{B}$  be observables for some quantum system. In particular, let  $\hat{A}$  and  $\hat{B}$  each be Hermitian operators with complete orthonormal (CON) sets of eigenkets,  $\{|a_n\rangle : 1 \leq n < \infty\}$  and  $\{|b_n\rangle : 1 \leq n < \infty\}$ , and associated eigenvalues,  $\{a_n : 1 \leq n < \infty\}$  and  $\{b_n : 1 \leq n < \infty\}$ , respectively.

(a) The commutator of  $\hat{A}$  and  $\hat{B}$  is, by definition,

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}.$$

Show that  $\frac{1}{j} [\hat{A}, \hat{B}]$  is an Hermitian operator.

(b) Assume that these observables commute, i.e.,

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = 0,$$

and that the eigenvalues of  $\hat{A}$  are distinct, as are the eigenvalues of  $\hat{B}$ . Show that every eigenket of  $\hat{A}$  is also an eigenket of  $\hat{B}$  and that every eigenket of  $\hat{B}$  is also an eigenket of  $\hat{A}$ , i.e.,  $\hat{A}$  and  $\hat{B}$  have a common, CON set of eigenkets.

**Problem 3.5**

Here we introduce the notation of tensor products, to permit us to deal with multiple quantum systems. Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be the Hilbert spaces of possible states for two quantum systems,  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , respectively. If we are interested in making a joint measurement on these two systems, e.g., the sum of their “positions”, etc., we need to have a way to describe states and observables for the joint system. Let  $\{|\phi_n\rangle_1 : 1 \leq n < \infty\}$  and  $\{|\theta_m\rangle_2 : 1 \leq m < \infty\}$  be orthonormal bases for  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, where the subscripts 1 and 2 indicate to which Hilbert space the states belong. The Hilbert space of states for the *joint* quantum system—i.e., systems 1 and 2 together—is spanned by the tensor product states  $\{|\phi_n\rangle_1 \otimes |\theta_m\rangle_2 : 1 \leq n, m < \infty\}$ , i.e., an arbitrary state  $|\psi\rangle \in \mathcal{H}$  can be expressed as a linear combination,

$$|\psi\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} (|\phi_n\rangle_1 \otimes |\theta_m\rangle_2), \quad (3)$$

by appropriate choice of the coefficients  $\{c_{nm}\}$ . Thus, because the inner product between  $|\phi_n\rangle_1 \otimes |\theta_m\rangle_2$  and  $|\phi_k\rangle_1 \otimes |\theta_l\rangle_2$  is defined to be,

$$({}_2\langle\theta_l| \otimes {}_1\langle\phi_k|)(|\phi_n\rangle_1 \otimes |\theta_m\rangle_2) = ({}_2\langle\theta_l|\theta_m\rangle_2)({}_1\langle\phi_k|\phi_n\rangle_1),$$

the inner product between  $|\psi\rangle$  from Eq. (3) and

$$|\psi'\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm} (|\phi_n\rangle_1 \otimes |\theta_m\rangle_2),$$

is

$$\langle\psi'|\psi\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm}^* c_{nm}.$$

- (a) Let  $\hat{A}_1$  be an observable of system 1, i.e., an Hermitian operator on  $\mathcal{H}_1$  with a complete set of eigenkets, and let  $\hat{B}_2$  be an observable of system 2, i.e., an Hermitian operator on  $\mathcal{H}_2$  with a complete set of eigenkets. The tensor product  $\hat{C} = \hat{A}_1 \otimes \hat{B}_2$  is a linear operator that maps the state  $|\phi_n\rangle_1 \otimes |\theta_m\rangle_2$  into the state  $(\hat{A}_1|\phi_n\rangle_1) \otimes (\hat{B}_2|\theta_m\rangle_2)$ .

Show that  $\hat{C}$  is an Hermitian operator on  $\mathcal{H}$  which has a complete set of eigenkets, so that  $\hat{C}$  is an observable on the joint Hilbert space of systems 1 and 2.

(b) Let

$$\hat{A}_1 = \sum_{n=1}^{\infty} a_n |a_n\rangle_{11} \langle a_n| \quad \text{and} \quad \hat{B}_2 = \sum_{m=1}^{\infty} b_m |b_m\rangle_{22} \langle b_m|$$

be the diagonal (eigenvalue/eigenket) decompositions of  $\hat{A}_1$  and  $\hat{B}_2$ , where the  $\{a_n\}$  are assumed to be distinct, as are the  $\{b_m\}$ . When we measure  $\hat{A}_1$  on system 1 *and* we measure  $\hat{B}_2$  on system 2 with the joint system being in state  $|\psi\rangle$ , given by Eq. (3), the outcome will be an ordered pair  $\{(a_n, b_m)\}$  of eigenvalues. The probability that  $(a_n, b_m)$  occurs is given by,

$$\Pr(a_n, b_m) = |\langle \psi | (|a_n\rangle_1 \otimes |b_m\rangle_2) |^2.$$

Show that this is a proper probability distribution. Express the marginal probabilities,  $\Pr(a_n)$  and  $\Pr(b_m)$ , in terms of  $|\psi\rangle$ , the  $\{|a_n\rangle_1\}$  and the  $\{|b_m\rangle_2\}$ .

(c) Specialize the results of (b) to the case of a product state, viz., a state that satisfies  $|\psi\rangle = |\psi_1\rangle_1 \otimes |\psi_2\rangle_2$ .

### Problem 3.6

Here we prove that it is impossible to clone the unknown state of a quantum system by means of a unitary evolution. It is a proof by contradiction. Suppose that we have a quantum system whose Hilbert space of states is  $\mathcal{H}_S$ , where  $S$  indicates that this is the *source* system. Suppose too that we have a *target* system whose Hilbert space of states is  $\mathcal{H}_T$ . We will assume that these two Hilbert spaces have the same dimensionality, e.g., 2.

We wish to construct a perfect cloner, viz., a unitary operator,  $\hat{U}$ , on the tensor product space  $\mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_T$  such that

$$\hat{U}(|\psi\rangle_S \otimes |0\rangle_T) = |\psi\rangle_S \otimes |\psi\rangle_T, \quad (4)$$

where  $|\psi\rangle_S$  is an *arbitrary* unit-length ket in  $\mathcal{H}_S$ , and  $|0\rangle_T$  is a reference (“blank”) unit-length ket in  $\mathcal{H}_T$ . Thus, the perfect cloner does not disturb the source state while it turns the target’s “blank” state into a clone of the source state.

Let  $|\psi_1\rangle_S$  and  $|\psi_2\rangle_S$  be two distinct, unit-length kets in  $\mathcal{H}_S$ , let  $\alpha$  and  $\beta$  be two non-zero complex numbers, and assume that we have found a perfect cloner operator  $\hat{U}$  that satisfies Eq. (4) for all unit-length source kets.

(a) Define

$$|\psi'\rangle_S = \frac{\alpha|\psi_1\rangle_S + \beta|\psi_2\rangle_S}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta \langle \psi_1 | \psi_2 \rangle_S]}}.$$

Use unitarity to evaluate the length of the ket  $|\theta\rangle \equiv \hat{U}(|\psi'\rangle_S \otimes |0\rangle_T)$ .

(b) Use the linearity of  $\hat{U}$  to show that

$$|\theta\rangle = \alpha'(|\psi_1\rangle_S \otimes |\psi_1\rangle_T) + \beta'(|\psi_2\rangle_S \otimes |\psi_2\rangle_T). \quad (5)$$

where

$$\alpha' \equiv \frac{\alpha}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta\langle\psi_1|\psi_2\rangle_S]}}$$
$$\beta' \equiv \frac{\beta}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta\langle\psi_1|\psi_2\rangle_S]}}$$

(c) Use Eq. (5) to evaluate the length of  $|\theta\rangle$ . Show that this result contradicts what you found in (a), and thus conclude that there is no unitary  $\hat{U}$  that can be a perfect cloner.