REVIEW OF NYQUIST CRITERION

Consider the modulated waveform $u(t) = \sum_k u_k p(t - kT)$**. The receiver filters** $u(t)$ **with** $q(t)$ **to receive** $r(t) = \sum_k u_k g(t - kT)$ **where** $\mathbf{g = p * q}$**.**

The composite filter g is ideal Nyquist if $g(kT) = 1$ **for** $k = 0$ **and 0 for** $k \in \mathbb{Z} \neq 0$**.**

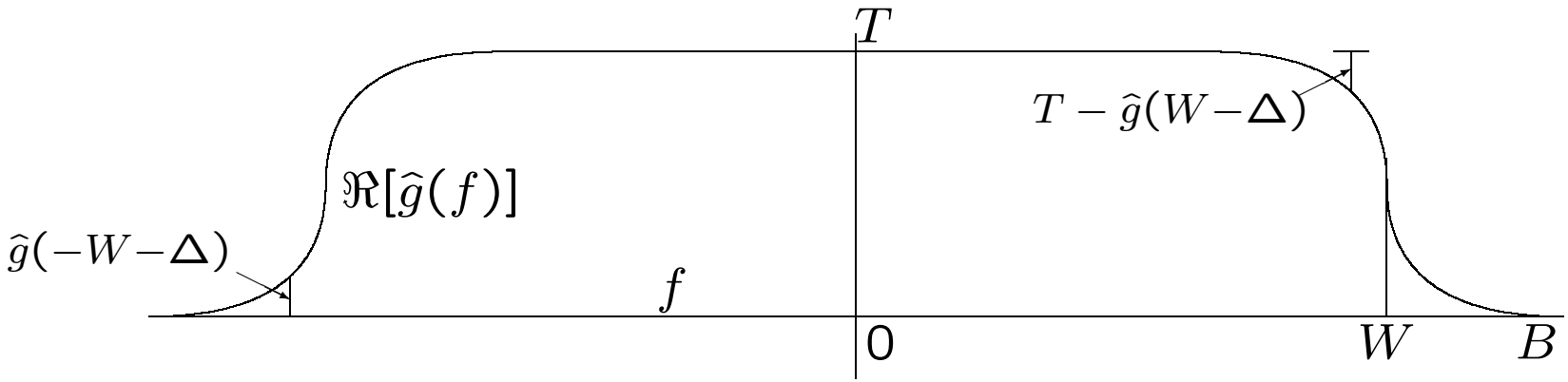
The T**-spaced samples of r then reproduce** $\{u_k\}$ **without intersymbol interference.**

Nyquist criterion: $g(t)$ **is ideal Nyquist iff**

$$\sum_m \widehat{g}(f + m/T)\,\mathsf{rect}(fT) = T\,\mathsf{rect}(fT)$$

Nyquist band (nominal band) is $W = 1/(2T)$.

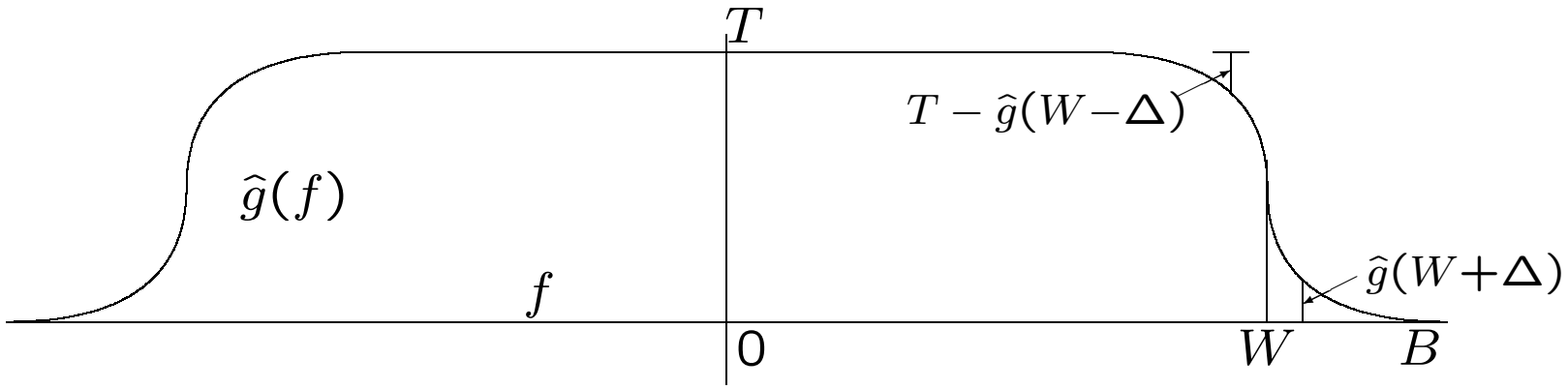
Actual baseband limit B should be close to W; assume $B < 2W$.



Tradeoff: want $\hat{g}(f) \approx \mathrm{rect}(f/(2W))$ but smooth.

Choose $\hat{g}(f)$ real and symmetric; then $g(t)$ is also real and symmetric.

Nyquist criterion for $g(t)$ real and symmetric is then a band-edge symmetry requirement.



$\Re\{\widehat{g}(f)\}$ **must satisfy the band edge symmetry condition to meet the Nyquist criterion.**

Choosing $\Im\{\widehat{g}(f)\} \neq 0$ **simply increases the energy outside of the Nyquist band with little effect on delay.**

Thus we restrict $\widehat{g}(f)$ **to be real (as in the raised cosine pulses used in practice).**

Because of noise, we choose $|\widehat{p}(f)| = |\widehat{q}(f)|$**.**

Since $\widehat{g}(f) = \widehat{p}(f)\widehat{q}(f)$**, this requires** $\widehat{q}(f) = \widehat{p}^*(f)$ **and thus** $q(t) = p^*(-t)$**. This means that**

$$g(t) = \int p(\tau)q(t-\tau)\,d\tau = \int p(\tau)p^*(\tau - t)\,d\tau$$

For $\hat{g}(f)$ real and satisfying Nyquist criterion,

$$g(kT) = \int p(\tau)p^*(\tau - kT)\,d\tau = \begin{cases} 1 & \textbf{for} \quad k = 0 \\ 0 & \textbf{for} \quad k \neq 0 \end{cases}$$

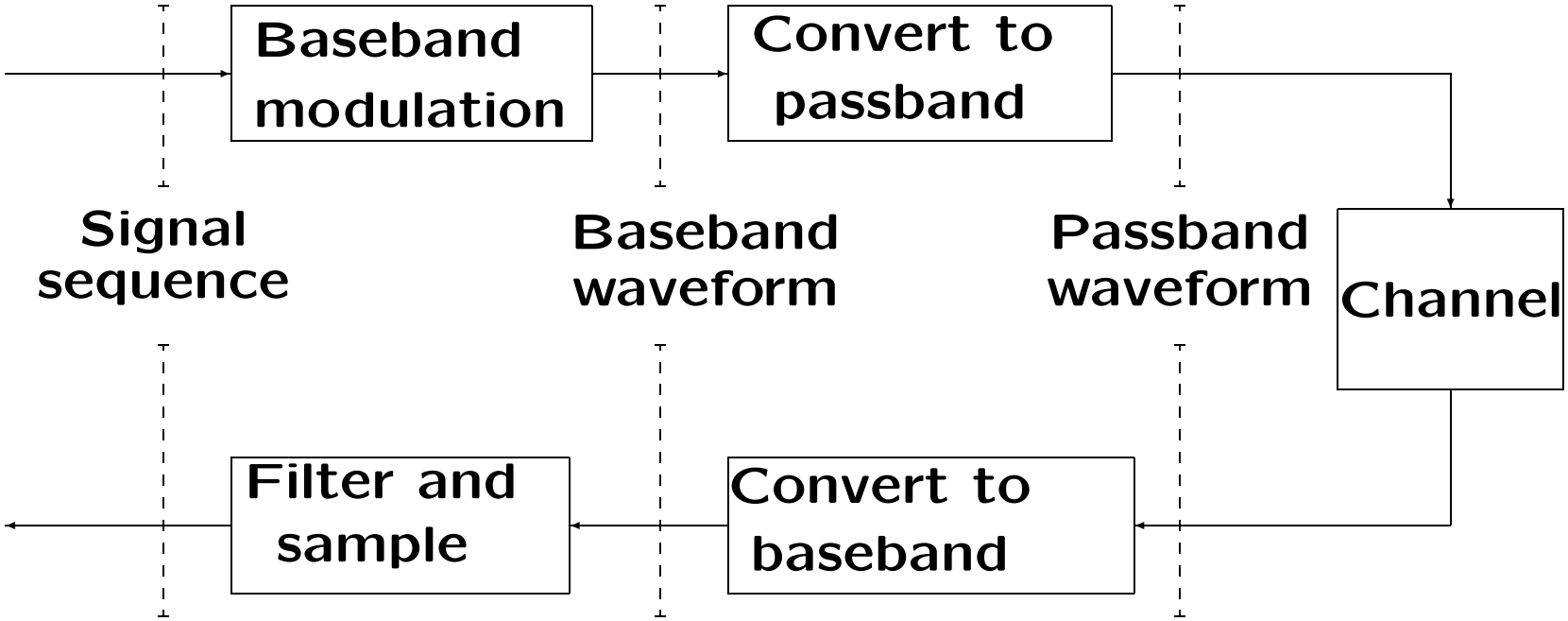
This means that $\{p(t-kT);\, k \in Z\}$ is an orthogonal set of functions.

These functions are usually real $\mathcal{L}_2$ functions, but might be complex.

Since $|\hat{p}(f)|^2 = \hat{g}(f)$, $p(t)$ is often called square root of Nyquist.

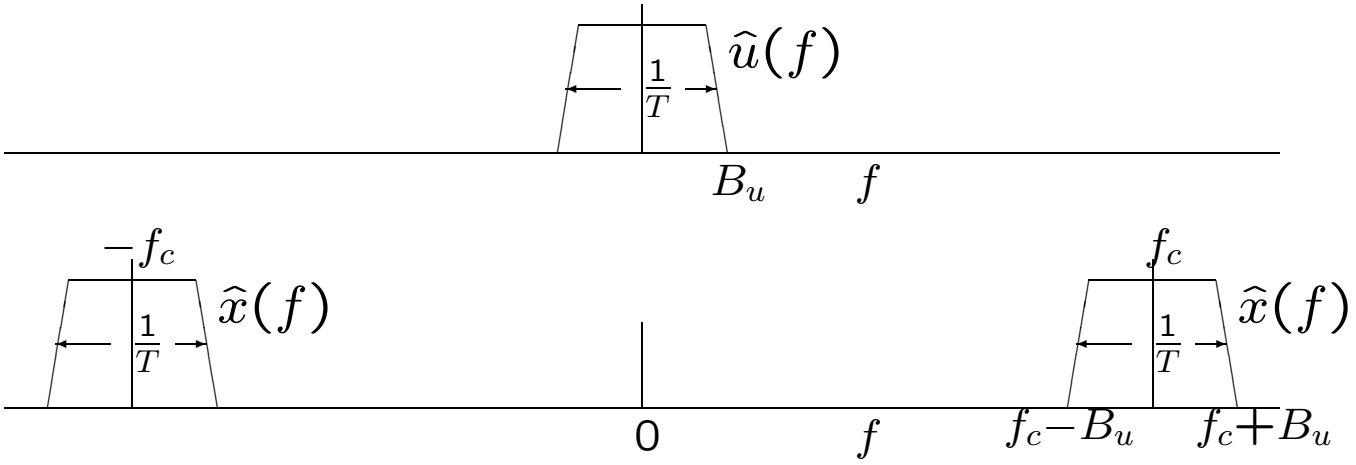
In vector terms, $\int u(\tau)q(kT-\tau)\,d\tau$ is the projection of u on $p(t-kT)$. $q(t)$ is called the matched filter to $p(t)$.

Frequency Translation (PAM and QAM)



PAM: $u(t)$ real

$$
\begin{aligned}
x(t) &= u(t)[e^{2\pi i f_c t} + e^{-2\pi i f_c t}] = 2u(t)\cos(2\pi f_c t), \\
\widehat{x}(f) &= \widehat{u}(f - f_c) + \widehat{u}(f + f_c).
\end{aligned}
$$



The bandwidth B is $2B_u$. The bandwidth is always the range of positive frequencies used.

QAM: $u(t)$ complex

QAM solves the frequency waste problem of DSB AM by using a complex baseband waveform $u(t)$.

$$x(t) = u(t)e^{2\pi i f_c t} + u^*(t)e^{-2\pi i f_c t}\,.$$

$$\begin{aligned} x(t) &= 2\Re\{u(t)e^{2\pi i f_c t}\} \\ &= 2\Re\{u(t)\}\cos(2\pi f_c t) - 2\Im\{u(t)\}\sin(2\pi f_c t)\,. \end{aligned}$$

It sends one baseband waveform on cos carrier, another on sine carrier.

Conceptually, QAM shifts complex $u(t)$ up by f_c. Then complex conjugate added to form real $x(t)$.

Think of shifting and conjugating separately.

$$u(t) \Longrightarrow u(t)e^{2\pi i f_c t} \Longrightarrow x(t).$$

If $B_u < f_c$, then $u(t)e^{2\pi i f_c t}$ is strictly in the positive frequency range. It can be perfectly filtered from $u^*(t)e^{2\pi i f_c t}$ at the receiver.

$$x(t) \Longrightarrow u(t)e^{2\pi i f_c t} \Longrightarrow u(t)$$

This filter is called a Hilbert filter (non-$\mathcal{L}_2$, non-practical, but useful conceptually).

COMPLEX (QAM) SIGNAL SET

R is input data rate in bits per second.

Segment b bits at a time ($M = 2^b$).

Map M symbols (binary b-tuples) to signal set.

Signal rate is $R_s = R/b$ signals per second.

$T = 1/R_s$ is the signal interval.

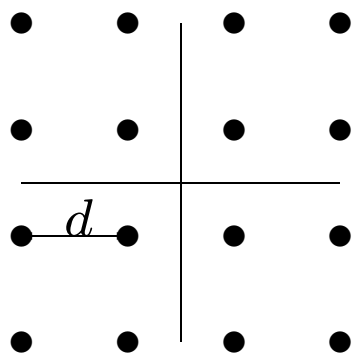
Signals $\{u_k\}$ are complex numbers (or real 2-tuples).

Signal set $\mathcal{A}$ is constellation of M complex numbers (or real 2-tuples)

A standard ($\sqrt{M} \times \sqrt{M}$)-QAM signal set is the Cartesian product of two $\sqrt{M}$-PAM sets; i.e.,

$$\mathcal{A} = \{(a' + ia'') \mid a' \in \mathcal{A}', a'' \in \mathcal{A}'\},$$

It is a square array of signal points located as below for $M = 16$.



The energy per 2D signal is

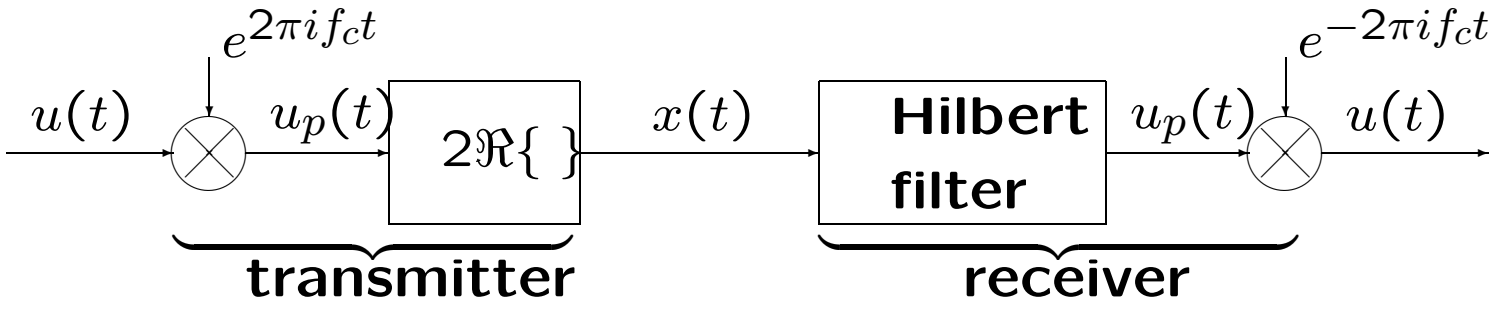
$$E_s = \frac{d^2[\sqrt{M}^2 - 1]}{6} = \frac{d^2[M-1]}{6}.$$

Choosing a good signal set is similar to choosing a 2D set of representation points in quantization.

Here one essentially wants to choose M points all at distance at least d so as to minimize the energy of the signal set.

This is even uglier than quantization. Try to choose the optimal set of 8 signals with $d = 1$.

For the most part, standard signal sets are used.



Note that $u(t)$ is complex, and viewed as vector in complex $\mathcal{L}_2$.

$x(t)$ is real and viewed as vector in real $\mathcal{L}_2$.

Orthogonal expansions must be treated with great care.

Above picture nice for analysis, but not usually so for implementation.

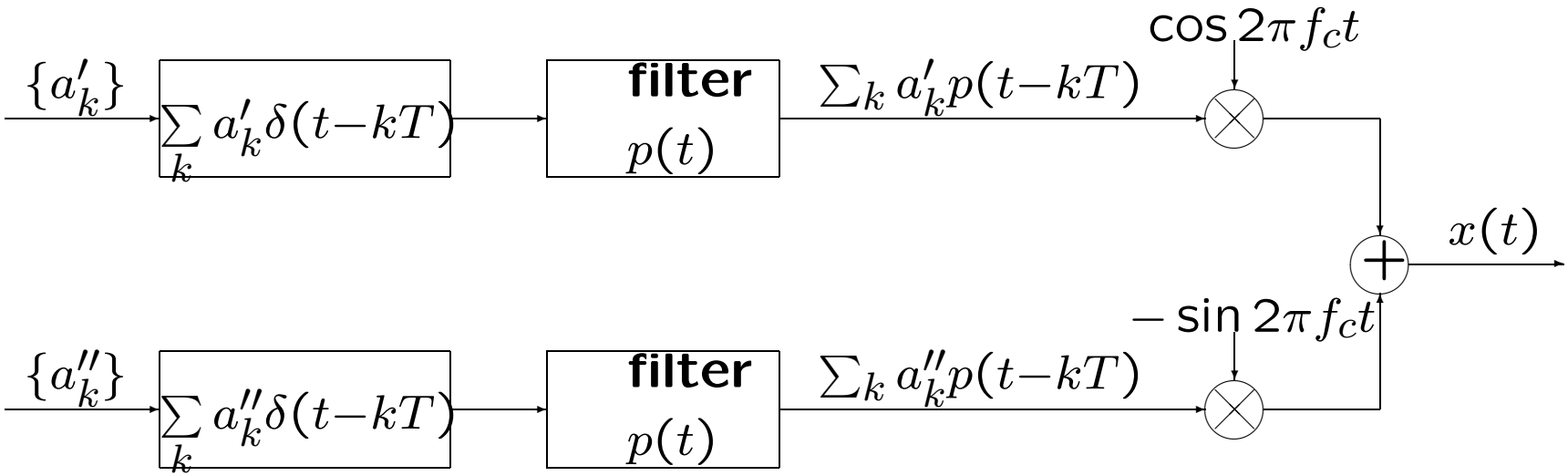
QAM IMPLEMENTATION (DSB-QC)

Assume $p(t)$ is real

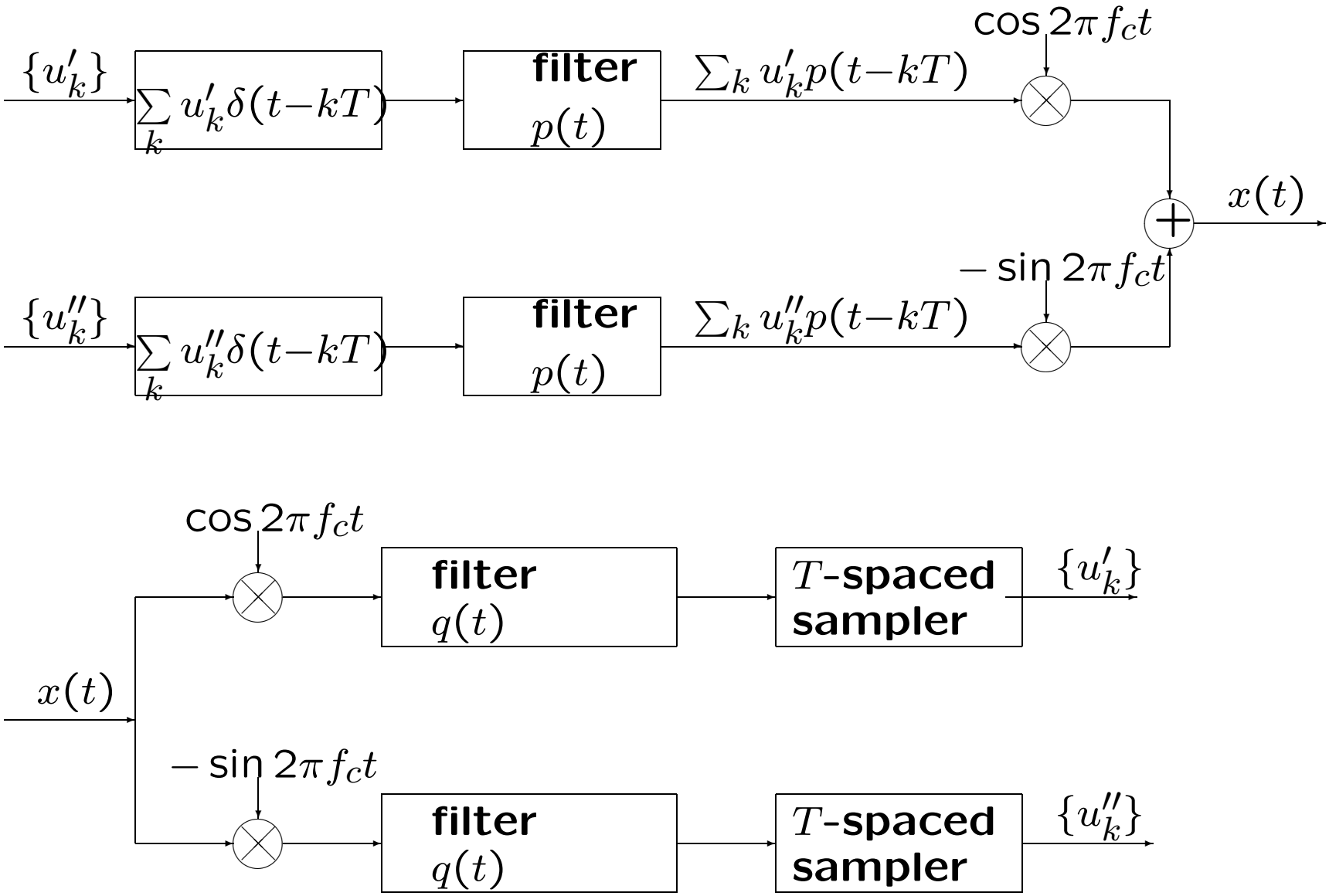
$$\begin{aligned}\Re\{u(t)\} &= \sum_k \Re\{u_k\}\, p(\frac{t}{T}-k),\\ \Im\{u(t)\} &= \sum_k \Im\{u_k\}\, p(\frac{t}{T}-k).\end{aligned}$$

With $u'_k = \Re\{u_k\}$ and $u''_k = \Im\{u_k\}$,

$$x(t) = 2\cos(2\pi f_c t)\left(\sum_k u'_k p(t-kT)\right) - 2\sin(2\pi f_c t)\left(\sum_k u''_k p(t-kT)\right)$$



Demodulate by multiplying $x(t)$ by both cosine and sign. Then filter out components around $2f_c$.


$\{u'_k\}$
$\sum_k u'_k \delta(t-kT)$
filter
$p(t)$
$\sum_k u'_k p(t-kT)$
$\cos 2\pi f_c t$
$x(t)$
$\{u''_k\}$
$\sum_k u''_k \delta(t-kT)$
filter
$p(t)$
$\sum_k u''_k p(t-kT)$
$-\sin 2\pi f_c t$
$\cos 2\pi f_c t$
$x(t)$
filter
$q(t)$
T-spaced
sampler
$\{u'_k\}$
$-\sin 2\pi f_c t$
filter
$q(t)$
T-spaced
sampler
$\{u''_k\}$

The DSB-QC implementation of QAM requires real filters p and q whose convolution g must satisfy Nyquist criterion.

A standard QAM signal set reduces the system to parallel PAM systems.

An arbitrary signal set can be used by combining the real and imaginary outputs.

Signal and noise around $2f_c$ must be filtered out before making baseband signal digital.

QAM, with sample spacing T, has baseband bandwidth $\frac{1}{2T}$ and passband bandwidth $\frac{1}{T}$

2 real degrees of freedom each T (each $1/W$). Over interval T_0 there are $2T_0/T = 2T_0W$ real degrees of freedom.

With PAM there are $2T_0W_0$ real degrees of freedom in baseband bandwidth W_0.

Break large baseband bandwidth W_0 into m passbands of width $W = W_0/m$.

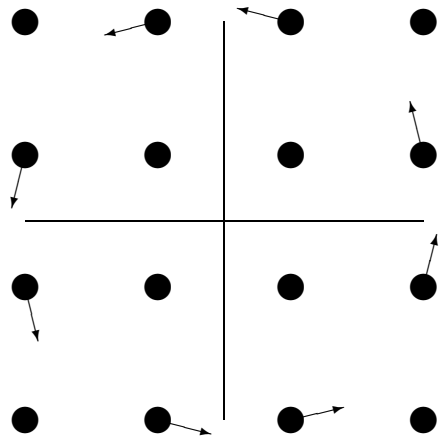
With QAM in each, m times $2T_0W$, i.e., $2T_0W_0$, real degrees of freedom overall.

QAM PHASE AND CARRIER RECOVERY

Let ϕ be the phase error at receiver, i.e., positive frequency waveform is $u_p(t) = u(t)e^{2\pi i f_c t}$.

Receiver maps to baseband with $e^{-2\pi i f_c t + i\phi}$.

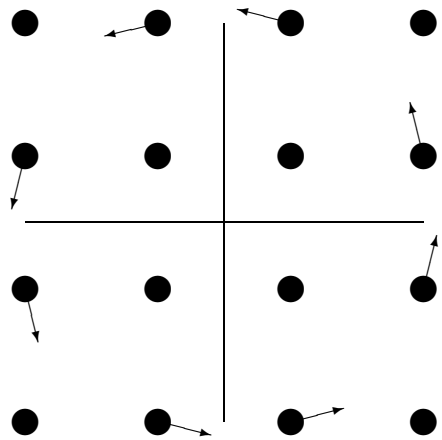
Baseband received waveform is $u(t)e^{i\phi}$.



$$r(kT) = e^{i\phi(kT)}u(kT)$$

Data point is rotated counter clockwise by ϕ.

ϕ can be corrected.



Phase error moves
Large points more.

Noise error moves
all points the same

Phase error changes slowly; its measurement is averaged and corrected over many intervals.

The noise is almost independent over time. It is detected as if phase error absent.

A phase error of $\pi/2$ can never be corrected by this method.

One approach to the uncorrectibility of large phase errors is to use differential phase transmission.

For 4-QAM, view as phase modulation. Let the signal map into phase changes instead of phase.

That is, $00 \rightarrow$ same signal as before; $01 \rightarrow$ add 90^o to phase ; etc.

For 16-QAM, differential phase can be used on quadrants.

Phase tracking can also sometimes be used to track frequency.

RANDOM PROCESSES

Sor far we have avoided random processes by looking only at random choices of signals and noise coefficients.

We converted the source waveform to a sequence, and said that only the probabilistic description of the sequence is relevant.

We related mean square error on the waveform to mean square error on the sequence, but usually just assumed the sequence to be iid.

Since sample sequences determine sample waveforms, there is merit to describing the sequence probabilistically.

ADDITIVE NOISE

Let $x(t)$ be the transmitted passband signal and $y(t) = x(t) + n(t)$ be the received passband signal.

We view $n(t)$ as a sample function of a random process $N(t)$.

We assume that a probabilistic description of $N(t)$ is known but that the sample function $n(t)$ is unknown.

$x(t)$ is known at the transmitter, but unknown at the receiver. From receiver point of view $x(t)$ is a sample function of a random process $X(t)$.

In terms of random processes, $Y(t) = X(t) + Z(t)$.

This implicitly assumes that the channel attenuation and delay are known and compensated.

It implicitly also means that $Z(t)$ is independent of $X(t)$.

These are standing assumptions until we start to study wireless systems.

A random process $\{Z(t)\}$ is a collection of rv's, one for each $t \in \mathbb{R}$.

For each epoch $t \in \mathbb{R}$, the rv $Z(t)$ is a function $Z(t, \omega)$ mapping sample points $\omega \in \Omega$ to real numbers.

For each $\omega \in \Omega$, $\{\mathbb{Z}(t, \omega\}$ is sample function $\{z(t)\}$.

A random process is defined by a rule establishing a joint density $f_{Z(t_1),\ldots,Z(t_k)}(z_1, \ldots, z_k)$ for all k, $t_1, \ldots, t_k$ and $z_1, \ldots, z_k$.

Our favorite way to do this is $\mathbf{Z}(t) = \sum Z_i \phi_i(t)$.

Joint densities on $Z_1, Z_2, \ldots$ define $\{\mathbf{Z}(t)\}$.

GAUSSIAN VARIABLES

Normalized Gaussian rv has density

$$f_N(n) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-n^2}{2}\right].$$

Arbitrary Grv Z is shift by $\overline{Z}$, scale by σ^2

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(z-\bar{Z})^2}{(2\sigma^2)}\right]$$

We refer to Z as $\mathcal{N}(\overline{Z}, \sigma^2)$

Gaussian rv's are important for the following reasons:

- The central limit theorem.
- Extremal properties
- Easy to manipulate analytically.
- Common models for noise

Refer to a k-tuple of rv's as $\mathrm{N} = \{N_1, \ldots, N_k\}$.

The set of k-tuples of rv's over a sample space is a vector space (but not the vector space $\mathbb{R}^{(k)}$ of real k-tuples).

Here we only want to use vector notation rather than any vector properties.

If $N_1, \ldots, N_k$ are iid $\mathcal{N}(0,1)$, then joint density is

$$\begin{aligned} f_{\mathbf{N}}(\mathbf{n}) &= \frac{1}{(2\pi)^{k/2}} \exp\left(\frac{-n_1^2 - n_2^2 - \cdots - n_k^2}{2}\right) \\ &= \frac{1}{(2\pi)^{k/2}} \exp\left(\frac{-\|\mathbf{n}\|^2}{2}\right). \end{aligned}$$

Note spherical symmetry.

A k-tuple of rv's is zero-mean jointly Gaussian if, for real a_{ij}, and for iid $\mathcal{N}(0,1)$ rv's $\{N_1, \ldots, N_m\}$,

$$Z_i = \sum_{j=1}^{m} a_{ij} N_j$$

i.e., Z is zero-mean jointly Gauss if $\mathrm{Z} = \mathbf{AN}$.

Jointly Gauss more than individually Gauss; must be linear combinations of iid Gauss.

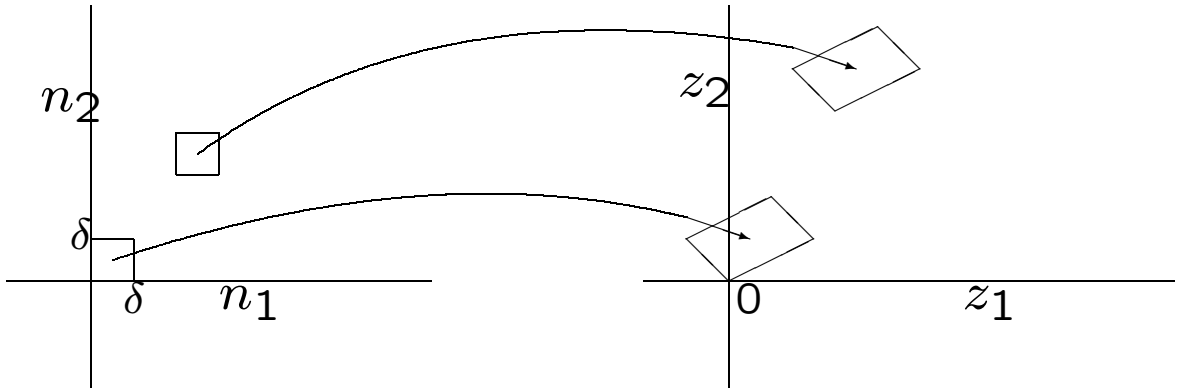
Jointly Gauss makes sense physically -

- **Rv's modelled as Gauss arise from CLT.**
- **Rv's modelled as Gauss are linear combinations of same underlying small variables.**

Think of z = An in terms of sample values and take $m = k$.

Ae$_i$ is mapped into ith column of A.

Thus unit cube is mapped into parallelepiped whose edges are the columns of A.



$Z_1 = N_1 + N_2$ **and** $Z_2 = N_1 + 2N_2$

The mapping n into $\mathbf{z} = \mathbf{An}$ maps a unit cube into a parallelepiped.

The volume of this parallelepiped is $|\det \mathbf{A}|$.

The density of $\mathbf{Z} = \mathbf{AN}$ at $\mathbf{z} = \mathbf{An}$ is scaled down from $f_{\mathbf{N}}$ by $|\det \mathbf{A}|$.

If A is singular, i.e., $\det \mathbf{A} = 0$, then density of Z doesn't exist.

We have seen that

$$f_{\mathbf{Z}}(\mathbf{An}) = \frac{f_{\mathbf{N}}(\mathbf{n})}{|\det \mathbf{A}|}.$$

Assume A non-singular. Then for all z,

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{f_{\mathbf{N}}(\mathbf{A}^{-1}\mathbf{z})}{|\det \mathbf{A}|}.$$

$$\begin{aligned} \mathbf{f}_{\mathbf{Z}}(\mathbf{z}) &= \frac{1}{(2\pi)^{k/2}|\det(\mathbf{A})|} \exp\left(\frac{-\|A^{-1}\mathbf{z}\|^2}{2}\right) \\ &= \frac{1}{(2\pi)^{k/2}|\det(\mathbf{A})|} \exp\left[-\frac{1}{2}\mathbf{z}^{\mathsf{T}}(\mathbf{A}^{-1})^{\mathsf{T}}\mathbf{A}^{-1}\mathbf{z}\right] \end{aligned}$$

For zero mean rv's, covariance of Z_1, Z_2 is $\mathsf{E}[Z_1 Z_2]$.

For k-tuple Z, covariance is matrix whose i, j element is $\mathsf{E}[Z_i Z_j]$. That is

$$\mathsf{K_Z} = \mathsf{E}[\mathsf{Z}\mathsf{Z}^\mathsf{T}].$$

For $\mathsf{Z} = \mathsf{AN}$, this becomes

$$\mathsf{K_Z} = \mathsf{E}[\mathsf{A}\mathsf{N}\mathsf{N}^\mathsf{T}\mathsf{A}^\mathsf{T}] = \mathsf{E}[\mathsf{A}\mathsf{A}^\mathsf{T}]$$

$$\mathsf{K_Z^{-1}} = \mathsf{E}[(\mathsf{A}^{-1})^\mathsf{T}\mathsf{A}^{-1}]$$

$$\mathsf{f_Z}(\mathsf{z}) = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\mathsf{K_Z})|}} \exp\left[-\frac{1}{2}\mathsf{z}^\mathsf{T}\mathsf{K_Z^{-1}}\mathsf{z}\right]$$

For $\mathbf{Z} = Z_1, Z_2$, let $\mathbf{E}[Z_1^2] = \mathbf{K}_{11} = \sigma_1^2$, $\mathbf{E}[Z_2^2] = \mathbf{K}_{11} = \sigma_2^2$. Let ρ be normalized covariance

$$\rho = \frac{\mathbf{E}[Z_1 Z_2]}{\sigma_1 \sigma_2} = \frac{\mathbf{k}_{12}}{\sigma_1 \sigma_2}.$$

$$\det(\mathbf{K_Z}) = \sigma_1^2 \sigma_2^2 - k_{12}^2 = \sigma_1^2 \sigma_2^2 (1 - \rho^2).$$

For A to be non-singular, we need $|\rho| < 1$. We then have

$$\mathbf{K}_{\mathbf{Z}^{-1}} == \frac{1}{1-\rho^2} \begin{bmatrix} 1/\sigma_1^2 & -\rho/(\sigma_1 \sigma_2) \\ -\rho/(\sigma_1 \sigma_2) & 1/\sigma_2^2 \end{bmatrix}$$

$$\mathbf{f_Z}(\mathbf{z}) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left(\frac{-\frac{z_1}{\sigma_1}^2 + 2\rho \frac{z_1}{\sigma_1}\frac{z_2}{\sigma_2} - \frac{z_2}{\sigma_2}^2}{2(1-\rho^2)} \right)$$

Lesson: Even for $k = 2$, this is messy.

6.450 Principles of Digital Communication I
Fall 2009