MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Fall 2008 | 6.436J/15.085J
Final exam, 1:30–4:30pm, (180 mins/100 pts) | 12/18/08

Whenever asked to explain or justify an answer, a formal proof is not needed, but just a brief explanation.

Problem 1: (30 points)
Consider the Markov chain shown in the figure. Each time that state i is visited, an independent random reward is obtained which is a normal random variable with mean i and variance 4. More precisely, the reward W_n obtained at time n, has a conditional PDF (given the past history), which is $N(i, 4)$.

(a) How many invariant distributions are there?

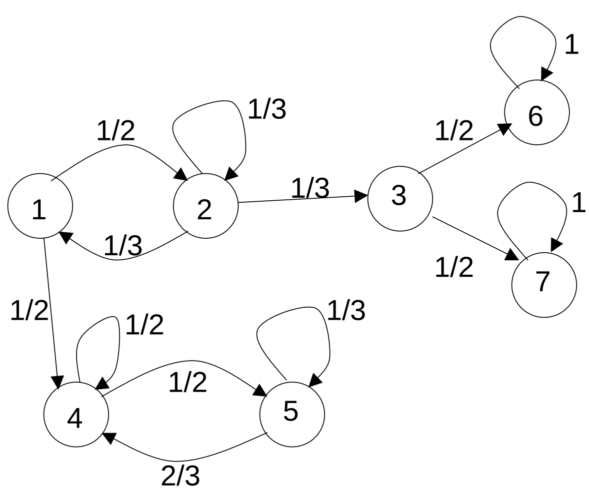
(b) Starting from state 1, what is the probability that the chain eventually visits state 5?

(c) Suppose that $X_0 = 4$. Does X_n converge almost surely? In distribution?

(d) Find $\mathbf{P}(X_n = 4 \mid X_{n+1} = 5,\ X_1 = 4)$, in the limit of very large n.

(e) Consider the average reward $R_n = (W_1 + \cdots + W_n)/n$. Conditioned on $X_0 = 3$, does R_n converge almost surely? If yes, to what? (A number or a random variable?) If not, explain why.

(f) Conditioned on $X_0 = 3$, what is the characteristic function of $W_1 + W_2$? (You do not need to do any algebra to simplify your answer.)



Solution:

(a) There are three independent invariant distributions, one for each recurrent class. Since any convex combination of these is an invariant distribution, there are infinitely many invariant distributions.

(b) The recursion equations are

$$\begin{aligned} p_1 &= \frac{1}{2} + \frac{1}{2}p_2 \\ p_2 &= \frac{1}{3}p_2 + \frac{1}{3}p_1 \end{aligned}$$

from which we get $p_1 = 2/3$.

(c) X_n does not converge almost surely, but in distribution it converges to the random variable which is 4 with probability π_4 and 5 with probability π_5. To compute these, we can argue

$$\pi_4 \frac{1}{2} = \pi_5 \frac{2}{3},$$

and with the additional equation $\pi_4 + \pi_5 = 1$, this gives $\pi_4 = 4/7, \pi_5 = 3/7$.

(d)

$$\mathbf{P}(X_n = 4 \mid X_{n+1} = 5,\, X_1 = 4) = \frac{P(X_n = 4, X_{n+1} = 5, \mid X_1 = 4)}{P(X_{n+1} = 5 \mid X_1 = 4)} = \frac{(1/2)\pi_4}{\pi_5} = \frac{2}{3}.$$

(e) Let R be the random variable which is 6 if $X_1 = 6$ and 7 if $X_1 = 7$. Then, the strong law of large numbers implies that R_n converges to R almost surely.

(f) With probability $1/2$, $W_1 + W_2$ is $N(12, 8)$ and with probability $1/2$ it is $N(14, 8)$. So,

$$\phi_{W_1+W_2}(t) = \frac{1}{2}e^{it12}e^{-4t^2} + \frac{1}{2}e^{it14}e^{-4t^2}.$$

Problem 2: (10 points)
A job takes an exponentially distributed amount of time to be processed, with parameter μ. While this job is being processed, new jobs arrive according to an independent Poisson process, with parameter λ. Find the PMF of the number of new jobs that arrive while the original job is being processed. (Justify your answer.)

Solution: Merge the arrival process and the original job process. The probability that k new jobs have arrived is the probability that the first k arrivals in the merged process come from the arrival process, and the $k+1$'st comes from the job process. So,

$$\mathbf{P}(k \text{ new jobs }) = (\frac{\lambda}{\lambda+\mu})^k \frac{\mu}{\lambda+\mu},\ k = 0, 1, 2, \ldots$$

Problem 3: (10 points)
A workstation consists of three machines, M_1, M_2, and M_3, each of which will fail after an amount of time T_i which is an independent exponentially distributed random variable, with parameter 1. Assume that the times to failure of the different machines are independent. The workstation fails as soon as **both** of the following have happened:
(i) Machine M_1 has failed;
(ii) At least one of the machines M_2 and M_3 has failed.

(a) Give a mathematical expression for the time of failure of the workstation in terms of the random variables T_i.

(b) Find the expected value of the time to failure of the workstation.

Solution: For part a,

$$\text{Failure time } = \max(T_1, \min(T_2, T_3)).$$

For part b, we have to wait an expected 1 time until M_1 fails. With probability $2/3$, M_1 fails after M_2 or M_3, so no more waiting is needed. With probability $1/3$, however, M_1 fails first and we have to wait until an arrival in the merged M_2, M_3 process which takes an expected value of $1/2$. So,

$$\mathbf{E}[\text{ Failture time }] = 1 + \frac{1}{3}\frac{1}{2} = \frac{7}{6}.$$

Problem 4: (10 points)
A fair six-sided die is tossed repeatedly and independently. Let $N_i(t)$ be the number of times a result of i appears in the first t tosses. We know that the joint PMF of the vector $N(t) = (N_1(t), \ldots, N_6(t))$ is multinomial.

(a) For $t > s$, find $\mathbf{E}[N_2(t) \mid N_1(s) = k]$.

(b) Find a and b such that $(N_1(t) - at)/b\sqrt{t}$ converges in distribution to a standard normal.

Solution: For part a, observe that the expected number of 2's in tosses $s+1, \ldots, t$ is $(t-s)/6$, and by conditioning, we can argue that the expected number of 2s in tosses $1, \ldots, s$ is $(s-k)/5$. So the final answer is $(t-s)/6 + (s-k)/5$.

For part b, we need to apply the central limit theorem. a needs to be the mean of $N_1(1)$, so $a = 1/6$. b needs to be the square root of the variance, $b = \sqrt{1/6 - (1/6)^2} = \frac{\sqrt{5}}{6}$.

Problem 5: (10 points)
Let X be a vector random variable with mean zero and covariance matrix V.

(a) Specify (in terms of V, and whenever possible) a square matrix U such that the covariance matrix of UX is the identity. State the conditions needed for this to be possible.

(b) Is it true that we can always find a matrix U (not necessarily square) so that the covariance of UX is the identity? Explain briefly.

Solution: We have that the covariance of UX is

$$\text{Cov}(UX) = UXX^T U = UVU,$$

so that if V is positive definite, we can just pick $U = V^{-1/2}$. Now for V to be positive definite, we must have that

$$a^T V a \neq 0,$$

for all $a \neq 0$ (since V is automatically nonnegative definite), which is

$$E[(a^T X)^2] \neq 0,$$

which is the the same as as requiring that $a^T X$ is not zero with probability 1. In summary, the condition for the existence of such a matrix is that the identically-zero random variable is not a linear combination of the random variables in X.

For part b, observe that its not possible to find such a matrix U if $X = 0$. On the other hand, if the vector X contains a random variable which is not identically 0, it is possible: we can just set $U = e_i^T$, where e_i is the i'th basis vector, and X_i is the random variable thats not identically 0.

Problem 6: (10 points)
Give an example of a sequence $\{X_n\}$ of r.v.s for which $\mathbf{E}[X_n^2] \to 0$, but X_n does not converge almost surely to 0.

Solution: Take X_n to be 1 with probability $1/n$ and 0 otherwise. Then, $E[X_n^2] = 1/n$ which goes to 0, but $X_n = 1$ infinitely often with probability 1 from the Borel-Cantelli lemma, so X_n does not converge to 0 almost everywhere.

Problem 7: (10 points)
Consider a sequence of events $\{A_n\}$ that satisfy $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$. However, the events are not independent, so that the Borel-Cantelli lemma does not apply. Instead, we have the following underlying structure. There is a sequence of independent random variables $\{X_n\}$ and a sequence of measurable functions $g_n : \mathbb{R}^2 \to \{0, 1\}$ such that $A_n = \{g_n(X_n, X_{n+1}) = 1\}$. Show that $\mathbf{P}(A_n \text{ i.o.}) = 1$.

Solution: At least one of

$$\sum_{n \text{ even}} \mathbf{P}(A_n), \quad \sum_{n \text{ odd}} \mathbf{P}(A_n),$$

must be infinite. Say it is the sum over even n that is infinite. Then, the events

$$A_2, A_4, A_6, \ldots$$

are all independent and by the Borel-Cantelli lemma, infinitely many of them must occur with probability one.

Problem 8: (10 points)
Let $\{X_n \mid n \geq 1\}$ be a Markov chain on the state space $\{1, \ldots, m\}$, for some integer m. Assume that this chain has a single recurrent class and no transient states.

(a) Let
$$M_n = \max_{i \leq n} X_i.$$
Is $\{M_n\}$ a Markov chain. If yes, give its one-step transition probabilities, and identify the transient and recurrent states. If not, explain why (briefly).

(b) Let $Y_n = (M_n, X_n)$. Is the process the process $\{Y_n\}$ a Markov chain? If yes, do not give a justification but give its one-step transition probabilities, and identify the transient and recurrent states. If not, explain why.

Solution:

(a) Not a Markov chain. Consider for example a particle at three states, $0, 1, 2$, which are connected as $1 - 0 - 2$ (i.e. there is a connection between 1 and 0 and between 0 and 2) . The particle jumps to a random neighbor with equal probability. The probability of transitions to $M_1 = 2$ from the history $M_0 = 1$ is 0, but the probability of transitions to $M_2 = 2$ from the history $M_0 = 1, M_1 = 1$ is strictly positive.

(b) Yes, this is a markov chain. The probability of transitioning from (M_1, i) to (M_2, j) is p_{ij} if one of the following two conditions holds:

- $\max(i, j) \leq M_1$ and $M_2 = M_1$.
- $j > M_1, j = M_2$.

and 0 otherwise. The recurrent state are the states $(m, i), i = 1, \ldots, m$; all other states are transient.