MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J | Fall 2008
Problem Set 11 | due 12/5/2008

Readings: Lecture notes/summaries and starred problems in the posted excerpt from the Markov Chapter of [BT].

Exercise 1. Let X_1, X_2, X_3 be independent exponential random variables with mean 1. Let

$$\alpha = \mathbb{P}(X_1 > X_2 + X_3).$$

(a) Find α, without calculating any integrals.

(b) Find the probability that the largest of the three random variables X_1, X_2, X_3 is larger than the sum of the other two. [You can express your answer in terms of the constant α from part (a).]

Exercise 2. Fast and slow customers arrive at a 24 hour store according to independent Poisson processes, each with rate 1 per minute. Fast customers stay in the bookstore for 1 minute, slow customers stay in the store for 2 minutes.

(a) What is the PMF of the total number of customer arrivals during a one minute interval?

(b) Find the variance of the number of customers in the store at 3 p.m.

(c) At 3 p.m., there is only one customer present in the store.

  (i) What is the probability, β, that the customer is a fast one?

  (ii) What is the PDF that this customer will depart before a new customer arrives? [You may express your answer in terms of the constant β from part (i). Also, you may leave your answer as a formula involving integrals – you do not have to evaluate the integrals.]

Let N_t be the number of fast customer arrivals during $[0, t]$.

(d) Does $(N_{2t} - N_t)/t$ converge in probability, as $t \to \infty$? With probability 1? If yes, to what? Outline a rigorous justification for your answers. You can start with t integer-valued and then argue for $t \in \Re$.

(e) Find (approximately) a time k such that

$$\mathbb{P}(N_k \geq 100) \approx 0.758.$$

Note that if Z is a standard normal random variable, then $\mathbb{P}(Z \leq 0.7) = 0.758$. [You do not need to be rigorous in deriving your answer. You may leave your answer in the form of an equation for k, which you do not need to solve numerically.]

Exercise 3. All ships travel at the same speed through a wide canal. Eastbound ships arrive as a Poisson process with an arrival rate of λ_E ships per day. Westbound ships arrive as an independent Poisson process with an arrival rate of λ_W ships per day. An indicator at a point in the canal is always pointing in the direction of travel of the most recent ship to pass. Each ship takes t days to traverse the canal.

(a) What is the probability that the next ship passing by the indicator causes it to change its direction?

(b) What is the probability that an eastbound ship will see no westbound ships during its eastward journey through the canal?

(c) If we begin observing at an arbitrary time, determine the probability mass function of the total number of ships we observe up to and including the seventh eastbound ship we see.

(d) If we begin observing at an arbitrary time, determine the PDF of the time until we see the seventh eastbound ship.

(e) Given that the pointer is pointing west:

  (i) What is the probability that the next ship to pass it will be westbound?

  (ii) What is the PDF of the remaining time until the pointer changes direction?

Exercise 4. Let Y be exponentially distributed with parameter λ_1. Let Z_k be Erlang of order k. with parameter λ_2. Assume that Y and Z_k are independent. Let $M_k = \max\{Y, Z_k\}$. Find a recursive formula for $\mathbb{E}[M_k]$, in terms of $\mathbb{E}[M_{k-1}]$. *Hint:* There is a similar problem in [BT].

Exercise 5. Let S be the set of arrival times in a Poisson process on $\mathbb{R}$ (i.e., a process that has been running forever), with rate λ. Each arrival time in S is displaced by a random amount. The random displacement associated with each element of S is a random variable that takes values in a finite set. We assume that the random displacements associated with different arrivals are independent and identically distributed. Show that the resulting process (i.e., the process whose arrival times are the displaced points) is a Poisson process with rate λ. (We expect a proof consisting of a verbal argument, using known properties of Poisson processes; formulas are not needed.)

Exercise 6. Consider a discrete-time, finite-state Markov chain $\{X_t\}$, with states $\{1, \dots, m\}$, and transition probabilities p_{ij}. States 1 and n are absorbing, that is, $p_{11} = 1$ and $p_{mm} = 1$. All other states are transient. Let A be the event that the state eventually becomes 1. For any possible starting state i, let $a_i = \mathbb{P}(A \mid X_0 = i)$ and assume that $a_i > 0$ for every $i \neq n$. Conditional on the information that event A occurs, is the process X_n necessarily Markov? If yes, provide a proof, together with a formula for its transition probabilities. If not, provide a counterexample.

6.436J / 15.085J Fundamentals of Probability
Fall 2008