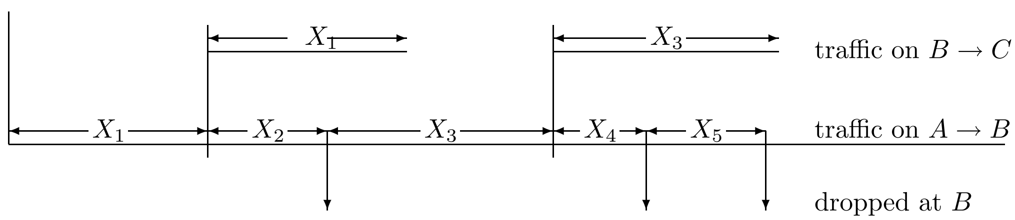
6.262 Discrete Stochastic Processes Monday April 4, 7:00-9:30pm, 2011
MIT, Fall 2011

Solutions to Quiz

Problem 1: An infinite sequence of packets are waiting to be sent, one after the other, from point A to an intermediate point B and then on to C. The lengths of the packets are IID rv's and thus X_i, the time intervals required to send successive packets from A to B are IID rv's assumed here to be exponential of rate λ. Thus the arrival epochs of packets at point b form a Poisson process of rate λ.

At point B, the receiver for the $A \to B$ link, after completely receiving a packet, passes it to the transmitter for the $B \to C$ link to send on to point C. The designer of point B had never taken 6.262 and knew nothing about queueing. Thus, instead of providing storage for packets, any packet received on link $A \to B$ is dropped if the transmitter for the $B \to C$ link is currently transmitting an earlier packet. The time to send an undropped packet on the $B \to C$ link is the same as that required for that packet on the $A \to B$ link.



1a) Let D_n be the event that packet n is dropped. Find $\Pr\{D_2\}$. Hint: Express this event as a relationship between X_1 and X_2.

Solution: Packet 2 is dropped if and only if (iff) $X_2 < X_1$ (It is not specified what happens if $X_2 = X_1$, but this is an event of 0 probability). That is, packet 2 starts transmission at A at the same time as packet 1 starts from B. Packet 2 arrives at B while packet 1 is still in transmission (and thus is dropped) iff $X_2 < X_1$, . Since X_1 and X_2 are IID, $\Pr\{X_2 < X_1\} = \Pr\{X_1 < X_2\}$, so $\Pr\{X_2 < X_1\} = 1/2$. Thus $\Pr\{D_2\} = 1/2$.

1b) For $n > 2$, find the probability that packets 1 through n are all successful, *i.e.*, $\Pr\{\bigcap_{i=1}^{n} D_i^c\}$.

Solution: The same idea works here as in part a). Packets 1 to n are all successful if $X_1 < X_2 < X_3 < \cdots < X_n$. That is, each successful packet starts transmission from B to C at the same instant as the next packet starts from A to B. Thus that next packet is not dropped if it completes arrival at B after the previous packet completes service at B.

Since $X_1, X_2, \dots, X_n$ are IID and, with probability 1, no two are equal, we see that each of the $n!$ possible orderings of $X_1, \dots, X_n$ are equiprobable. Thus the probability that all are successful is $\Pr\{X_1 < X_2 < \cdots < X_n\} = 1/n!$

1c) For $n > 2$, find the probability that packets 2 through n are all dropped, *i.e.*, $\Pr\{\bigcap_{i=2}^{n} D_i\}$. Hint: Parts b) and c) are solved in very different ways

Solution: For packets 2 through n to all be dropped, we must have $X_2 + X_3 + \cdots + X_n < X_1$, *i.e.*, all these packets must arrive at B while packet 1 is still in transmission. The elegant way to find this is to use the memoryless property of the exponential. That is, given $X_1 > X_2$, (an event of probability 1/2), $X_1 - X_2$ is exponential and IID with X_3. Thus the probability that $X_3 < X_1 - X_2$, conditional on $X_1 > X_2$, is 1/2, so the marginal probability that X_2 and X_3 are dropped is 1/4. In the same way, for each i, conditional on $X_1 > X_2 + \cdots X_i$, the probability that $X_{i+1} > X_1 - X_2 - \cdots - X_i$ is 1/2. Thus, the marginal probability that $X_2, \ldots, X_n$ are all dropped is 2^{n-1}.

A slightly more abstract but simpler way to see this is to visualize X_1 as the first arrival of one Poisson process and $X_2, X_3, \ldots$ as the arrivals of a second Poisson process. Viewing these as the splitting of a Poisson process with binary equiprobable switching, we see that the probability that the first $n-1$ arrivals come from the $X_2, \ldots$ process is the probability that the switch selects the $X_2, \ldots$ process $n-1$ times, *i.e.*, $1/2^{n-1}$.

1d) Find $\Pr\{D_3 \mid D_2\}, \Pr\{D_3 \mid D_2^{\mathsf{c}}\}$, and $\Pr\{D_3\}$.

Solution: From the solution to c), we see that $\Pr\{D_3 \mid D_2\}$, (*i.e.*, the probability that packet 3 is dropped given that 2 is dropped) is 1/2. From part b), $\Pr\{D_3^{\mathsf{c}} D_2^{\mathsf{c}}\} = 1/6$. Thus $\Pr\{D_3^{\mathsf{c}} \mid D_2^{\mathsf{c}}\} = 1/3$ and $\Pr\{D_3 \mid D_2^{\mathsf{c}}\} = 2/3$. To see this latter result in a different way, given D_2^{c}, we can order X_1, X_2, X_3 in 3 equiprobable ways: $X_3 < X_1 < X_2$ or $X_1 < X_3 < X_2$ or $X_1 < X_2 < X_3$. Packet 3 is dropped in the first two of these. Finally,

$$\Pr\{D_3\} = \frac{1}{2}\Pr\{D_3 \mid D_2\} + \frac{1}{2}\Pr\{D_3 \mid D_2^{\mathsf{c}}\} = \frac{7}{12}$$

1e) Find the distribution of the kth idle period on the link $B \to C$.

Solution: At the end of a transmission on link $B \to C$, the link is idle until the next arrival at B. It is almost sufficient to say that the Poisson process of arrivals at B implies that starting at any given time, the time until the next arrival is exponential with rate λ. The problem with this argument is that the end of a transmission on link $B \to C$ is a function of the Poisson process from A to B. Thus we have to argue that if packet n is not dropped, then while it is being transmitted to C, $X_{n+1}, X_{n+2}, \ldots.$ are the following interarrival times to B and these are independent of X_n.

1f) Does the sequence of beginning busy periods on the link $B \to C$ constitute the sequence of renewal epochs of a renewal process?

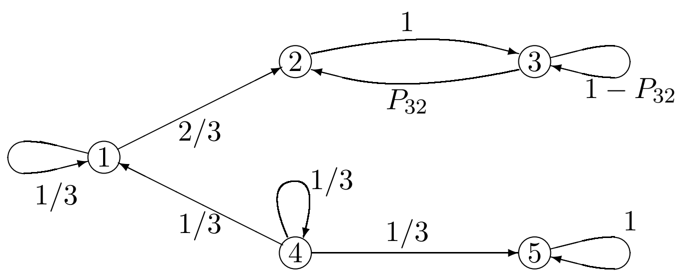
Solution: A simple and perfectly correct answer is that this is not a renewal process because $\mathsf{f}_{X_1}(x) = \lambda e^{-\lambda x}$ whereas the following inter-renewal interval is the sum of a transmission followed by an idle period of density $\lambda e^{-\lambda x}$.

A more subtle question is whether this is a delayed renewal process, *i.e.*, whether subsequent renewals after the first are IID. The answer is again no. To see this, let us compare the length Y_n of the nth packet from B to C given, first, that the length Y_{n-1} of the $n-1$st packet on $B \to C$ is very small and second that Y_{n-1} is very large. If Y_{n-1} is abnormally small, then it is highly likely that no packet was dropped between the two transmissions. Given this condition, $Y_n = Y_{n-1} + Z_n$ where Z_n is exponential of rate λ. If Y_{n-1} is very large, on the other hand, it is highly likely for intermediate packets to be dropped. Given this condition, Y_n is the sum of two terms, first, the exponential time from when that packet starts (on $A \to B$) to when packet $n-1$ finishes on $B \to C$, and, second, the additional time until that packet is fully received at B. Thus the expected length of Y_n under this condition is $2/\lambda$.

1g) Now suppose that $\{X_n;\, n \geq 1\}$ are IID and continuous but not exponential. Re-solve parts a) and b) for this case.

Solution: The same ordering argument as in parts a) and b) works. Thus the probability that packets $2, \dots, n$ are not dropped is $1/n!$ for all $n \geq 2$.

Problem 2: Consider the followng finite-state Markov chain.



2a) Identify the transient states and identify each class of recurrent states.

Solution: States 1 and 4 are transient. States 2 and 3 constitute a class of recurrent states and state 5 constitues another class (a singleton class) of recurrent states.

2b) For each recurrent class, find the steady-state probability vector $\boldsymbol{\pi} = (\pi_1, \dots, \pi_5)$ for that class.

Solution: Let $\boldsymbol{\pi}^{(1)}$ be the steady-state vector for class $\{2, 3\}$ and let $\boldsymbol{\pi}^{(2)}$ be the steady state vector for class $\{5\}$. For the class $\{5\}$, $\boldsymbol{\pi}^{(2)} = (0, 0, 0, 0, 1)$.

For the class $\{2, 3\}$, the steady-state equations (written just for the recurrent class) are

$$\pi_3^{(1)} P_{32} = \pi_2^{(1)}; \qquad \pi_2^{(1)} P_{23} + \pi_3^{(1)} P_{33} = \pi_3^{(1)}; \qquad \pi_2^{(1)} + \pi_3^{(1)} = 1$$

From the first and third of these, we get $\pi_3^{(1)} = 1/(P_{32}+1)$ and $\pi_2^{(1)} = P_{32}/(P_{32}+1)$

2c) Find the following n-step transition probabilities, $P_{ij}^n = \Pr\{X_n = j \mid X_0 = i\}$ as a function of n. Give a brief explanation of each (no equations are required).

i) P_{44}^n

ii) P_{45}^n

iii) P_{41}^n

iv) $P_{43}^n + P_{42}^n$

v) $\lim_{n\to\infty} P_{43}^n$

Solution:

i) $P_{44}^n = 3^{-n}$ since n successive self-loop transitions, each of probability $1/3$, are required.

ii) $P_{45}^n = (1/3) + (1/3)^2 + \cdots (1/3)^n = \frac{1}{2}(1 - 3^{-n})$. The reason for this is that there are n walks going from 4 to 5 in n steps; each such walk contains the 4 to 5 transition at a different time. If it occurs at time i then there are $i - 1$ self-transitions from 4 to 4, so the probability of that walk is $(1/3)^i$. As a check, note that $\lim_{n\to\infty} P_{45}^n = 1/2$ which can be verified from the symmetry in leaving the transient state, node 4.

iii) $P_{41}^n = n(1/3)^n$ since there are n walks that go from 4 to 1 in n steps, one for each step in which the $4 \to 1$ transition can be made. Each walk has probability $(1/3)^n$.

iv) $P_{43}^n + P_{42}^n \;=\; 1 - P_{44}^n - P_{45}^n - P_{41}^n \;=\; \frac{1}{2} - \frac{2n+1}{2}3^{-n}$ (since $\sum_j P_{4,j}^n = 1$).

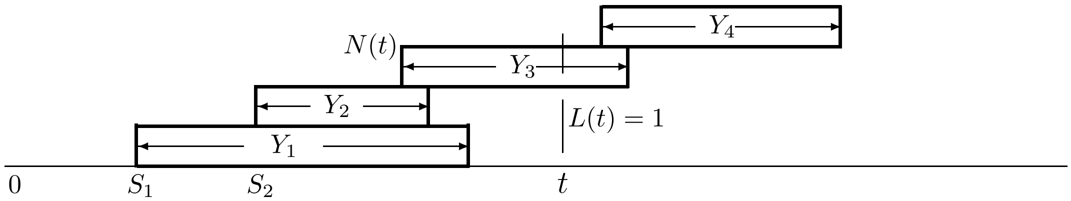
v) From iv), note that $\lim_{n\to\infty} P_{43}^n + P_{42}^n = 1/2$, which can again be verified by the symmetry at node 4. Given that $X_n \in \{2, 3\}$,

$$\lim_{m\to\infty} \Pr\{X_m = 3 \mid X_n \in \{2,3\}\} = \pi_3^{(1)} = \frac{1}{(P_{32} + 1)}$$

Since $\lim_{n\to\infty} \Pr\{X_n \in \{2,3\} \mid X_0 = 4\} = \frac{1}{2}$, we see that

$$\lim_{m\to\infty} P_{43}^m = \frac{1}{2(P_{32} + 1)}$$

Problem 3: Consider a (G/G/∞) 'queueing' system. That is, the arriving customers form a renewal process, *i.e.*, the interarrival intervals $\{X_n;\, n \geq 1\}$ are IID. You may assume throughout that $\mathsf{E}[X] < \infty$. Each arrival immediately enters service; there are infinitely many servers, so one is immediately available for each arrival. The service time Y_i of the ith customer is a rv of expected value $\bar{Y} < \infty$ and is IID with all other service times and independent of all inter-arrival times. There is no queueing in such a system, and one can easily intuit that the number in service never becomes infinite since there are always available servers.



3a) Give a simple example of distributions for X and Y in which this system never becomes empty. Hint: Deterministic rv's are fair game.

Solution: Suppose, for example that the arrivals are deterministic, one arrival each second. Suppose that each service takes 2 second. Then when the second arrival occurs, the first service is only half over. When the third arrival arrives, the first service is just finishing, but the second arrival is only half finished, etc. Thus the system never becomes empty (although it is empty initially from 0 top S_1).

This also illustrates the unusual behavior of systems with an unlimited number of servers. If the arrival rate is increased, there are simply more servers at work simultaneously; there is no possibility of the system becoming overloaded.

A general version of the example above is any distributions for X and Y such that a number a exists for which $X < a$ WP1 and $Y > a$ WP1. This is necessary and sufficient (except possible fussing about whether the system is empty at some instant where a departure and an arrival are simultaneous, which is a silly thing to fuss about).

3b) We want to prove Little's theorem for this type of system, but there are no renewal instants for the entire system. As illustrated above, let $N(t)$ be the renewal counting process for the arriving customers and let $L(t)$ be the number of customers in the system (*i.e.*, receiving service) at time t. In distinction to our usual view of queueing systems, assume that there is no arrival at time 0 and the first arrival occurs at time $S_1 = X_1$. The nth arrival occurs at time $S_n = X_1 + \cdots + X_n$.

Carefully explain why, for each sample point ω and each time $t > 0$,

$$\int_0^t L(\tau, \omega)\, d\tau \;\le\; \sum_{i=1}^{N(t,\omega)} Y_i(\omega)$$

Solution: For the case in the above figure, $\int_0^t L(\tau, \omega)\, d\tau$ is Y_1 plus Y_2 plus the part of Y_3 on which service has been completed by t. Thus $\int_0^t L(\tau, \omega)\; d\tau \le Y_1 + Y_2 + Y_3$. Since $N(t) = 3$ for this example, $\int_0^t L(\tau, \omega)\, d\tau \le \sum_{i=1}^{N(t)} Y_i$.

In general, $\sum_{i=1}^{N(t)} Y_i$ is the total amount of service required by all customers arriving before or at t. $\int_0^t L(\tau, \omega)\, d\tau$ is the total amount of service already provided by t. Since not all the required service has necessarily been provided by t (for example in the sample path of the figure, customer 3 arrives before t but is not completely served by t), the inequality is as given.

3c) Find the limit as $t \to \infty$ of $\frac{1}{t} \sum_{i=1}^{N(t,\omega)} Y_i(\omega)$ and show that this limit exists WP1.

Solution: This is similar to a number of limits we have taken, but we give a more careful derivation here since many of you came quite close to a rigorous solution. The trick, which most of you saw, is to first multiply and divide by $N(t,\omega)$ and later to use the SLLN and the strong law for renewals, *i.e.*,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N(t,\omega)} Y_i(\omega) = \lim_{t\to\infty} \frac{N(t,\omega)}{t} \frac{\sum_{i=1}^{N(t,\omega)} Y_i(\omega)}{N(t,\omega)} \tag{1}$$

The derivation from here will consist of the 4 parts listed below:

(i) Let $A_1 = \{\omega : \lim_{t\to\infty} N(t,\omega)/t = 1/\bar{X}\}$. Then $\Pr\{A_1\} = 1$. This is the strong law for renewal processes applied to the arrival process.

(ii) Let $A_2 = \{\omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i(\omega) = \bar{Y}\}$. Then $\Pr\{A_2\} = 1$. This is the SLLN applied to the service-time sequence.

(iii) For any given $\omega \in A_1 \bigcap A_2$,

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t,\omega)} Y_i(\omega)}{N(t,\omega)} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i(\omega) \tag{2}$$

This might seem obvious, but but it really should be proven. From the convergence in (ii), we know that for any $\varepsilon > 0$, there is an n_0 (depending on ω) such that $|\frac{1}{n}\sum_{i=1}^{n} Y_i(\omega) - \bar{Y}| < \varepsilon$ for all $n \geq n_0$. From part (i), for this same ω, there is a t_0 such that $N(t,\omega) \geq n_0$ for all $t \geq t_0$. Thus

$$\left| \frac{\sum_{i=1}^{N(t,\omega)} Y_i(\omega)}{N(t,\omega)} - \bar{Y} \right| < \varepsilon \qquad \text{for all } t \geq t_0$$

This establishes the existence of the left hand limit in (2) and also shows that the limit is $\bar{Y}$.

(iv) Let $f(t)$ and $g(t)$ be functions $\mathbb{R} \to \mathbb{R}$ and assume that $\lim_{t\to\infty} f(t) = a$ and $\lim_{t\to\infty} g(t) = b$ where a and b are finite. Then $\lim_{t\to\infty} f(t)g(t) = ab$. We essentially stated this in lecture, and it follows from the identity

$$(f(t) - a)(g(t) - b) = f(t)g(t) - ab - a(g(t) - b) - b(f(t) - a)$$

Then for any $\varepsilon > 0$, choose t_0 such that $|f(t) - a| < \varepsilon$ and $|g(t) - b| < \varepsilon$ for all $t \geq t_0$. Then $|f(t)g(t) - ab| < \varepsilon^2 + \varepsilon(|a| + |b|)$, so it follows that $\lim_t f(t)g(t) = ab$.

Finally, taking f and g as the first and second term respectively in (1) and using the limits in parts (i) and (iii), we see that (1) is satisfied for all $\omega \in A_1 \bigcap A_2$, *i.e.*, for a set of probability 1.

3d) Assume that the service time distribution is bounded between 0 and some $b > 0$, *i.e.*, that $\mathsf{F}_Y(b) = 1$. Carefully explain why

$$\int_0^{t+b} L(\tau,\omega)\, d\tau \;\geq\; \sum_{i=1}^{N(t,\omega)} Y_i(\omega)$$

Solution: This is almost the same as part b). All customers that have entered service by t have completed service by $t+b$, and thus the difference between the two terms above is the service provided to customers that have arrived between t and $t+b$.

3e) Find the limit as $t \to \infty$ of $\frac{1}{t}\int_0^t L(\tau,\omega)\,d\tau$ and indicate why this limit exists WP1.

Solution: Since the bound in part d) is valid for all t, we can replace t by $t-b$ for all $t > b$, getting

$$\int_0^t L(\tau,\omega)\,d\tau \;\geq\; \sum_{i=1}^{N(t-b,\omega)} Y_i(\omega)$$

Combining this with the result of b) and dividing by t,

$$\frac{1}{t}\sum_{i=1}^{N(t-b,\omega)} Y_i(\omega) \leq \frac{1}{t}\int_0^t L(\tau,\omega)\,d\tau \;\leq \frac{1}{t}\sum_{i=1}^{N(t,\omega)} Y_i(\omega)$$

From part c), the limit as $t \to \infty$ of the right side of this is $\bar{Y}/\bar{X}$ for the set $A_1 \bigcap A_2$. This also shows that for the same set of ω,

$$\lim_{t\to\infty}\frac{1}{t-b}\sum_{i=1}^{N(t-b,\omega)} Y_i(\omega) = \frac{\bar{Y}}{\bar{X}} \qquad \text{and thus}$$

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{N(t-b,\omega)} Y_i(\omega) = \frac{\bar{Y}}{\bar{X}}$$

Since $\frac{1}{t}\int_0^t L(\tau,\omega)\,d\tau$ is sandwiched between two terms that are both $\bar{Y}/\bar{X}$ in the limit $t \to \infty$ for all $\omega \in A_1 \bigcap A_2$, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t L(\tau)\,d\tau = \frac{\bar{Y}}{\bar{X}} \qquad \text{WP1}$$

This is Little's theorem for the G/G/∞ case.

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