6.262 Discrete Stochastic Processes — Wednesday, May 18, 9:00-12:00 noon, 2011
MIT, Spring 2011

Final examination

There are 4 questions, each with several parts. We attempted to put the easier parts of each question toward the beginning of that question.

You have 3 hours to finish the final. If any part of any question is unclear, please ask.

The blue books are for scratch paper only. Please put your answers and an explanation of your reasoning in the white books.

Please put your name on each white booklet you turn in.

The questions have been designed to require very little computation if done in the most insightful way. Explore the question carefully and think about the simplest cases before lauching into a messy solution.

Partial credit: Careful reasoning will receive generous partial credit even if the final answer is incorrect, and correct answers with incorrect or inadequate reasons will not receive full credit.

We will try to match your grade on a question to the level of understanding, both intuitive and mathematical, that you have exhibited rather than your ability to manipulate a large number of equations.

Your explanations need not appear to be rigorous, but should be adequate as a full explanation to a fellow student, leaving nothing in doubt.

Problem 1: A final exam is started at time 0 for a class of n students. Each student is allowed to work until completing the exam. It is known that for $1 \leq i \leq n$, student i's time to complete the exam, X_i, is exponentially distributed with density $\mathsf{f}_X(x) = \lambda e^{-\lambda x}$; $x \geq 0$. The times $X_1, \dots, X_n$ are IID.

a) Let Z be the time at which the last student finishes. Show that Z has a distribution function $\mathsf{F}_Z(z)$ given by $[1 - \exp(-\lambda t]^n$.

b) Let T_1 be the time at which the first student leaves. Show that the probability density of T_1 is given by $n\lambda e^{-n\lambda t}$. For each i, $2 \leq i \leq n$, let T_i be the interval from the departure of the $i-1$st student to that of the ith. Show that the density of each T_i is exponential and find the parameter of that exponential density. Explain why each T_i is independent. Finally note that $Z = \sum_{i=1}^{n} T_i$.

Problem 2: (The results from problem 1 will be useful here). A Yule process is a continuous time version of a branching process with the special property that the process never decreases. The process starts at time 0 with one organism. This organism splits into two organisms after a time Y_1 with the density $\mathsf{f}_{Y_1}(y) = \lambda e^{-\lambda y}$, $y \geq 0$. Each of these two organisms splits into two more organisms after independent exponentially-distributed delays, each with this same density $\lambda e^{-\lambda y}$. In general, each old and new organism continues to split forever after a delay y with this same density $\lambda \exp(-\lambda y)$.

a) Let T_1 be the time at which the first organism splits, and for each $i > 1$, let T_i be the interval from the $i-1$st splitting until the ith. Show that T_i is exponential with parameter $i\lambda$ and explain why the T_i are independent.

b) For each $n \geq 1$, let the continuous rv S_n be the time at which the nth splitting occurs, *i.e.*, $S_n = T_1 + \cdots + T_n$. Find a simple expression for the distribution function of S_n. Hint: look carefully at the solution to parts a) and b) of problem 1.

c) Let $X(t)$ be the number of organisms at time $t > 0$. Express the distribution function of $X(t)$ for each $t > 0$ in terms of S_n for each n. Show that $X(t)$ is a rv for each $t > 0$ (*i.e.*, show that $X(t)$ is finite WP1).

d) Find $\mathsf{E}[X(t)]$ for each $t > 0$.

e) Is $\{X(t);\, t \geq 0\}$ a countable-state Markov process? Explain carefully. If so describe the embedded Markov chain and identify each state as positive recurrent, null recurrent, or transient.

f) Is $\{X(t);\, t \geq 0\}$ an irreducible countable-state Markov process?

g) Is $\{S_n;\, n \geq 1\}$ either a martingale or a submartingale?

h) Now suppose the births of a Yule process (viewing the original organism as being born at time 0) constitute the input to a queueing process (each birth enters the queue at its time of birth). The queue has a single server and whenever n organisms are in the system (queue plus server), the time to completion of service for the given organism is exponential with rate μ_n. This means that if a new organism enters the queue while the server is busy, the service rate changes according to the new number in the system. Let $Z(t)$ be the number in the system at time t.

h(i): Is $\{Z(t);\, t \geq 0\}$ a countable-state Markov process?

h(ii): Is $Z(t)$ a rv for each $t > 0$ no matter what the μ_n are?

Problem 3: A random walk $\{S_n;\, n \geq 1\}$, with $S_n = \sum_{i=1}^{n} X_i$, has the following probability density for each X_i

$$\mathsf{f}_X(x) = \begin{cases} \frac{e^{-x}}{e-e^{-1}} \ ; & -1 \leq x \leq 1 \\ \\ = 0 \ ; & \text{elsewhere.} \end{cases}$$

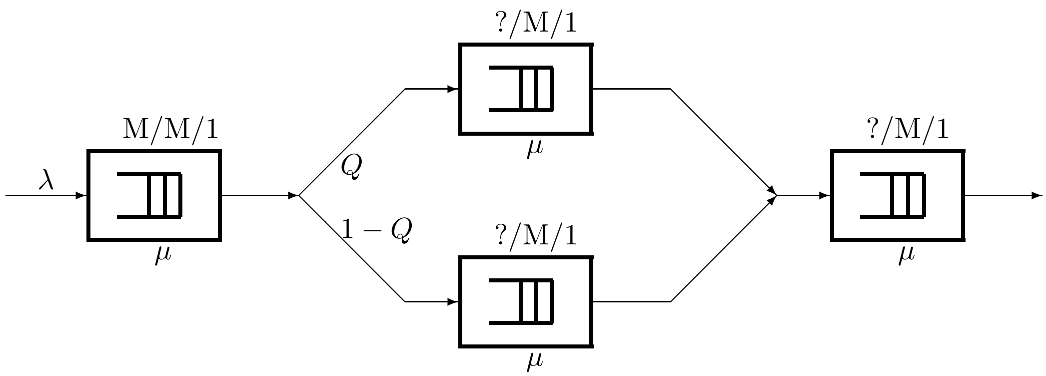
a) Find the values of r for which $g(r) = \mathsf{E}\left[\exp(rX)\right] = 1$. Hint: these values turn out to be integers.

b) Let P_α be the probability that the random walk ever crosses a threshold at some given $\alpha > 0$. Use the Wald identity to find an upper bound to P_α of the form $P_\alpha \leq e^{-\alpha A}$ where A is a constant that does not depend on α. Evaluate A. Hint: you may assume that the Wald identity applies to a single threshold at $\alpha > 0$ without any lower threshold or assume another threshold at some $\beta < 0$.

c) Use the Wald identity to find a lower bound to P_α of the form $P_\alpha \geq Be^{-\alpha A}$ where A is the same as in part b) and $B > 0$ is a constant that does not depend on α. Hint: Keep it simple — you are not being asked to find the tightest possible such bound. If you use 2 thresholds, find your lower bound in the limit $\beta \to \infty$.

Problem 4: A queueing system has four queues in the configuration shown. Each queue is identical, with a single server with IID exponential service times, each with density $\mu e^{-\mu x}$. The service times are IID both within each queue and between each queue.

The left most queue (queue 1, say) is $M/M/1$ with an input which is a Poisson process of rate $\lambda < \mu$. Assume that this process started at time $-\infty$. The inputs to the other queues are indicated in the figure below. More specifically, each output from queue 1 is switched to one of the intermediate queues, say queues 2 and 3. Each output goes to queue 2 with probability Q and to queue 3 with probabiity 1-Q. These switching decisions are independent of the inputs and outputs from queue 1. The outputs from queues 2 and 3 are them combined and pass into queue 4.



a) Describe the output process from queue 1. That is, describe whether it is a renewal process, and if so, whether it is a Poisson process. At what rate do customers leave queue 1? What can you say about the relation between the outputs and the inputs of queue 1?

b) Describe the input processes to queues 2 and 3. Are they Poisson, and if so, of what rate? What is the relationship between the input process to queue 2 and that to queue 3?

c) Describe the output processes from queues 2 and 3. Are they Poisson, and what is the relationship between them? What is the rate at which customers leave each of these queues?

d) Describe the input and output process for queue 4 (whether they are Poisson, what their rates are, and whether they are independent)?

e) Find the expected delay through the entire system for those customers going through queue 2.