15.081J/6.251J Introduction to Mathematical Programming

Lecture 11: Duality Theory IV

1 Outline

Slide 1

- Overview and objectives
- Weistrass Theorem
- Separating hyperplanes theorem
- Farkas lemma revisited
- Duality theorem revisited

2 Overview and objectives

Slide 2

- So far: Simplex $\longrightarrow$ Duality $\longrightarrow$ Farkas lemma
- Disadvantages: specialized to LP, relied on a particular algorithm
- Plan today: Separation (A Geometric property) $\longrightarrow$ Farkas lemma $\longrightarrow$ Duality
- Purely geometric, generalizes to general nonlinear problems, more fundamental

3 Closed sets

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- A set $S \subset \Re^n$ is closed if $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots$ is a sequence of elements of S that converges to some $\boldsymbol{x} \in \Re^n$, then $\boldsymbol{x} \in S$.
- Every polyhedron is closed.

4 Weierstrass' theorem

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If $f : \Re^n \mapsto \Re$ is a continuous function, and if S is a nonempty, closed, and bounded subset of $\Re^n$, then there exists some $\boldsymbol{x}^* \in S$ such that $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S$. Similarly, there exists some $\boldsymbol{y}^* \in S$ such that $f(\boldsymbol{y}^*) \geq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S$.
Note: Weierstrass' theorem is not valid if the set S is not closed. Consider, $S = \{x \in \Re \mid x > 0\}$, $f(x) = x$

5 Separation

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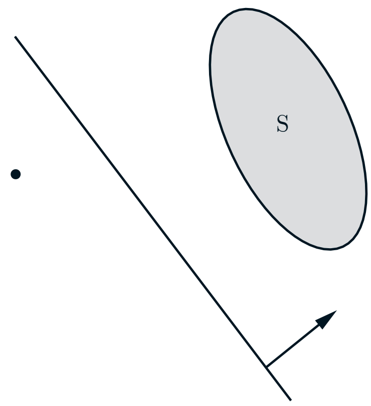
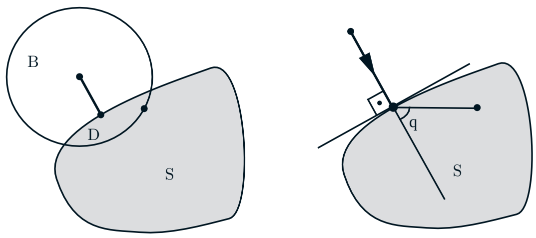
Theorem: Let S be a nonempty closed convex subset of $\Re^n$ and let $\boldsymbol{x}^* \in \Re^n$: $\boldsymbol{x}^* \notin S$. Then, there exists some vector $\boldsymbol{c} \in \Re^n$ such that $\boldsymbol{c}'\boldsymbol{x}^* < \boldsymbol{c}'\boldsymbol{x}$ for all $\boldsymbol{x} \in S$.





(a) (b)

5.1 Proof

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- Fix $\boldsymbol{w} \in S$
- $B = \left\{ \boldsymbol{x} \;\middle|\; ||\boldsymbol{x} - \boldsymbol{x}^*|| \leq ||\boldsymbol{w} - \boldsymbol{x}^*|| \right\}$,
- $D = S \cap B$
- $D \neq \emptyset$, closed and bounded. Why?
- Consider min $||\boldsymbol{x} - \boldsymbol{x}^*||$

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- By Weierstrass' theorem there exists some $\boldsymbol{y} \in D$ such that

$$||\boldsymbol{y} - \boldsymbol{x}^*|| \leq ||\boldsymbol{x} - \boldsymbol{x}^*||, \qquad \forall\ \boldsymbol{x} \in D.$$

- $\forall \boldsymbol{x} \in S$ and $\boldsymbol{x} \notin D$, $||\boldsymbol{x} - \boldsymbol{x}^*|| > ||\boldsymbol{w} - \boldsymbol{x}^*|| \geq ||\boldsymbol{y} - \boldsymbol{x}^*||$.
- $\boldsymbol{y}$ minimizes $||\boldsymbol{x} - \boldsymbol{x}^*||$ $\forall \boldsymbol{x} \in S$.
- Let $\boldsymbol{c} = \boldsymbol{y} - \boldsymbol{x}^*$

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- $\boldsymbol{x} \in S$. $\forall \lambda$ satisfying $0 < \lambda \leq 1$, $\boldsymbol{y} + \lambda(\boldsymbol{x} - \boldsymbol{y}) \in S$ (S convex)
- $||\boldsymbol{y} - \boldsymbol{x}^*||^2 \leq ||\boldsymbol{y} + \lambda(\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{x}^*||^2$

$$= ||\boldsymbol{y} - \boldsymbol{x}^*||^2 + 2\lambda(\boldsymbol{y} - \boldsymbol{x}^*)'(\boldsymbol{x} - \boldsymbol{y}) + \lambda^2||\boldsymbol{x} - \boldsymbol{y}||^2$$

- $2\lambda(\boldsymbol{y} - \boldsymbol{x}^*)'(\boldsymbol{x} - \boldsymbol{y}) + \lambda^2||\boldsymbol{x} - \boldsymbol{y}||^2 \geq 0$.
- Divide by λ, $(\boldsymbol{y} - \boldsymbol{x}^*)'(\boldsymbol{x} - \boldsymbol{y}) \geq 0$, i.e.,

$$\begin{aligned} (\boldsymbol{y} - \boldsymbol{x}^*)'\boldsymbol{x} &\geq (\boldsymbol{y} - \boldsymbol{x}^*)'\boldsymbol{y} \\ &= (\boldsymbol{y} - \boldsymbol{x}^*)'\boldsymbol{x}^* + (\boldsymbol{y} - \boldsymbol{x}^*)'(\boldsymbol{y} - \boldsymbol{x}^*) \\ &> (\boldsymbol{y} - \boldsymbol{x}^*)'\boldsymbol{x}^*. \end{aligned}$$

- $\boldsymbol{c} = \boldsymbol{y} - \boldsymbol{x}^*$ proves theorem

6 Farkas' lemma

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Theorem: If $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is infeasible, then there exists a vector $\boldsymbol{p}$ such that $\boldsymbol{p}'\boldsymbol{A} \geq \boldsymbol{0}'$ and $\boldsymbol{p}'\boldsymbol{b} < 0$.

- $S = \left\{\boldsymbol{y} \;\middle|\; \text{there exists } \boldsymbol{x} \text{ such that } \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x},\ \boldsymbol{x} \geq \boldsymbol{0}\right\}$ $\boldsymbol{b} \notin S$.
- S is convex; nonempty; closed;
  S is the projection of $\{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ onto the $\boldsymbol{y}$ coordinates, is itself a polyhedron and is therefore closed.
- $\boldsymbol{b} \notin S$: $\exists \boldsymbol{p}$ such that $\boldsymbol{p}'\boldsymbol{b} < \boldsymbol{p}'\boldsymbol{y}$ for every $\boldsymbol{y} \in S$.
- Since $\boldsymbol{0} \in S$, we must have $\boldsymbol{p}'\boldsymbol{b} < 0$.
- $\forall \boldsymbol{A}_i$ and $\forall \lambda > 0$, $\lambda \boldsymbol{A}_i \in S$ and $\boldsymbol{p}'\boldsymbol{b} < \lambda \boldsymbol{p}'\boldsymbol{A}_i$
- Divide by λ and then take limit as λ tends to infinity: $\boldsymbol{p}'\boldsymbol{A}_i \geq 0 \Rightarrow \boldsymbol{p}'\boldsymbol{A} \geq \boldsymbol{0}'$

7 Duality theorem

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$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{array} \qquad\qquad \begin{array}{ll} \max & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} = \boldsymbol{c}' \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

and we assume that the primal has an optimal solution $\boldsymbol{x}^*$. We will show that the dual problem also has a feasible solution with the same cost. Strong duality follows then from weak duality.

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- $I = \{i \mid \boldsymbol{a}_i'\boldsymbol{x}^* = b_i\}$
- We next show: if $\boldsymbol{a}_i'\boldsymbol{d} \geq 0$ for every $i \in I$, then $\boldsymbol{c}'\boldsymbol{d} \geq 0$

- $\boldsymbol{a}_i'(\boldsymbol{x}^* + \epsilon \boldsymbol{d}) \geq \boldsymbol{a}_i \boldsymbol{x}^* = b_i$ for all $i \in I$.
- If $i \notin I$, $\boldsymbol{a}_i' \boldsymbol{x}^* > b_i$ hence $\boldsymbol{a}_i'(\boldsymbol{x}^* + \epsilon \boldsymbol{d}) > b_i$.
- $\boldsymbol{x}^* + \epsilon \boldsymbol{d}$ is feasible

- By optimality $\boldsymbol{x}^*$, $\boldsymbol{c}'\boldsymbol{d} \geq 0$
- By Farkas' lemma
$$\boldsymbol{c} = \sum_{i \in I} p_i \boldsymbol{a}_i.$$
- For $i \notin I$, we define $p_i = 0$, so $\boldsymbol{p}'\boldsymbol{A} = \boldsymbol{c}'$.
-
$$\boldsymbol{p}'\boldsymbol{b} = \sum_{i \in I} p_i b_i = \sum_{i \in I} p_i \boldsymbol{a}_i' \boldsymbol{x}^* = \boldsymbol{c}'\boldsymbol{x}^*,$$

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