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Lecture 12: Local Controllability[1]

In this lecture, nonlinear ODE models with an input are considered. Partial answers to the general controllability question (which states can be reached in given time from a given state by selecting appropriate time-dependent control action) are presented.

More precisely, we consider systems described by

$$\dot{x}(t) = a(x(t), u(t)), \tag{12.1}$$

where $a : \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n$ is a given continuously differentiable function, and $u = u(t)$ is an m-dimensional time-varying input to be chosen to steer the solution $x = x(t)$ in a desired direction. Let U be an open subset of $\mathbf{R}^n$, $\bar{x}_0 \in \mathbf{R}^n$. The *reachable set* for a given $T > 0$ the (U-locally) *reachable set* $R^U(\bar{x}_0, T)$ is defined as the set of all $x(T)$ where $x : [0, T] \mapsto \mathbf{R}^n$, $u : [0, T] \mapsto \mathbf{R}^m$ is a bounded solution of (12.1) such that $x(0) = \bar{x}_0$ and $x(t) \in U$ for all $t \in [0, T]$.

Our task is to find conditions under which $R^U(\bar{x}_0, T)$ is guaranteed to contain a neigborhood of some point in $\mathbf{R}^n$, or, alternatively, conditions which guarante that $R^U(\bar{x}_0, T)$ has an empty interior. In particular, when $\bar{x}_0$ is a controlled equilibrium of (12.1), i.e. $a(\bar{x}_0, \bar{u}_0) = 0$ for some $\bar{u}_0 \in \mathbf{R}^m$, complete local controllability of (12.1) at $\bar{x}_0$ means that for every $\epsilon > 0$ and $T > 0$ there exists $\delta > 0$ such that $R^U(\bar{x}, T) \supset B_\delta(\bar{x}_0)$ for every $\bar{x} \in B_\delta(\bar{x}_0)$, where $U = B_\epsilon(\bar{x}_0)$ and

$$B_r(\bar{x}) = \{\bar{x}_1 \in \mathbf{R}^n : \ |\bar{x}_1 - \bar{x}| \leq r\}$$

denotes the ball of radius r centered at $\bar{x}$.

[1] Version of October 31, 2003

12.1 Systems with controllable linearizations

A relatively straightforward case of local controllability analysis is defined by systems with controllable linearizations.

12.1.1 Controllability of linearized system

Let $x_0: \ [0,T] \mapsto \mathbf{R}^n$, $u_0: \ [0,T] \mapsto \mathbf{R}^m$ be a bounded solution of (12.1). The standard *linearization* of (12.1) around the solution $(x_0(\cdot), u_0(\cdot))$ describes the dependency of small state increments $\delta_x(t) = x(t) - x_0(t) + o(\delta_x(t))$ on small input increments $\delta_u(t) = u(t) - \delta_u(t)$:

$$\dot{\delta}_x(t) = A(t)\delta_x(t) + B(t)\delta_u(t), \tag{12.2}$$

where

$$A(t) = \left.\frac{da}{dx}\right|_{x=x_0(t),u=u_0(t)}, \quad B(t) = \left.\frac{da}{du}\right|_{x=x_0(t),u=u_0(t)} \tag{12.3}$$

are bounded measureable matrix-valued functions of time.

Let us call system (12.2) *controllable on time interval* $[0,T]$ if for every $\bar{\delta}_x^0, \bar{\delta}_x^T \in \mathbf{R}^n$ there exists a bounded measureable function $\delta_u: \ [0,T] \mapsto \mathbf{R}^m$ such that the solution of (12.2) with $\delta_x(0) = \bar{\delta}_x^0$ satisfies $\delta_x(T) = \bar{\delta}_x^T$. The following simple criterion of controllability is well known from the linear system theory.

Theorem 12.1 *System (12.2) is controllable on interval* $[0,T]$ *if and only if the matrix*

$$W_c = \int_0^T M(t)^{-1}B(t)B(t)'(M(t)')^{-1}dt$$

is positive definite, where $M = M(t)$ *is the evolution matrix of (12.2), defined by*

$$\dot{M}(t) = A(t)M(t), \quad M(0) = I.$$

Matrix W_c is frequently called the *Grammian*, or *Gram matrix* of (12.2) over $[0,T]$. It is easy to see why Theorem 12.1 is true: the variable change $\delta_x(t) = M(t)z(t)$ reduces (12.2) to

$$\dot{z}(t) = M(t)^{-1}B(t)\delta_u(t).$$

Moreover, since

$$z(T) = \int_0^T M(t)^{-1}B(t)\delta_u(t)dt$$

is a linear integral dependence, function δ_u can be chosen to belong to any subclass which is dense in $L^1(0,T)$. For example, $\delta_u(t)$ can be selected from the class of polynomials, class of piecewise constant functions, etc.

Note that controllability over an interval Δ implies controllability over every interval Δ_+ containing Δ, but in general does not imply controllability over all intervals Δ_- contained in Δ. Also, system (12.2) in which $A(t) = A_0$ and $B(t) = B_0$ are constant is *equivalent* to controllability of the pair (A,B).

12.1.2 Consequences of linearized controllability

Controllability of linearization implies local controllability. The converse is not true: a nonlinear system with an uncontrollable linearization can easily be controllable.

Theorem 12.2 *Let $a: \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n$ be continuously differentiable. Let $x_0: [0,T] \mapsto \mathbf{R}^n$, $u_0: [0,T] \mapsto \mathbf{R}^m$ be a bounded solution of (12.1). Assume that system (12.2), defined by (12.3), is controllable over $[0,T]$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $\bar{x}_0, \bar{x}_T$ satisfying*

$$|\bar{x}_0 - x_0(0)| < \delta, \quad |\bar{x}_T - x_0(T)| < \delta$$

there exist functions $x: [0,T] \mapsto \mathbf{R}^n$, $u: [0,T] \mapsto \mathbf{R}^m$ satisfying the ODE in (12.2) and conditions

$$x(0) = \bar{x}_0, \quad x(T) = \bar{x}_T, \; |x(t) - x_0(t)| < \epsilon, \quad |u(t) - u_0(t) < \epsilon \;\; \forall\, t \in [0,T].$$

In other words, if linearization around a trajectory (x_0, u_0) is controllable then from every point in a sufficiently small neigborhood of $x_0(0)$ the solution of (12.1) can be steered to every point in a sufficiently small neigborhood of $x_0(T)$ by applying a small perturbation $u = u(t)$ of the nominal control $u_0(t)$. In particular, this applies when $x_0(t) \equiv \bar{x}_0$, $u_0(t) \equiv \bar{u}_0$ is a conditional equilibrium, in which case A, B are constant, and hence controllability of (12.2) is easy to verify.

When system (12.2) is not controllable, system (12.1) could still be: for example, the second order ODE model

$$\begin{aligned} \dot{x}_1 &= x_2^3, \\ \dot{x}_2 &= u \end{aligned}$$

has an uncontrollable linearization around the equilibrium solution $x_1 \equiv 0$, $x_2 \equiv 0$, but is nevertheless locally controllable.

The proof of Theorem 12.2 is based on the implicit mapping theorem. Let $e_1, \dots, \epsilon_n$ be the standard basis in $\mathbf{R}^n$. Let $\delta_u = \delta_u^k$ be the controls which cause the solution of (12.2) wit $\delta_x(0) = 0$ to reach $\delta_x(T) = e_k$. For $\epsilon > 0$ let

$$B_\epsilon = \{\bar{x} \in \mathbf{R}^n : \; |\bar{x}| < \epsilon\}.$$

The function $S : B_\epsilon \times B_\epsilon \mapsto \mathbf{R}^n$, which maps $w = [w_1; w_2; \dots; w_n] \in B_\epsilon$ and $v \in B_\epsilon$ to $S(w,v) = x(T)$, where $x = x(t)$ is the solution of (12.1) with $x(0) = x_0(0) + v$ and

$$u(t) = u_0 + \sum_{k=1}^{n} w_k \delta_u^k(t),$$

is well defined and continuously differentiable when $\epsilon > 0$ is sufficiently small. The derivative of S with respect to w at $w = v = 0$ is identity. Hence, by the implicit mapping theorem, equation $S(w,v) = \bar{x}$ has a solution $w \approx 0$ whenever $|v|$ and $|\bar{x} - x_0(T)|$ are small enough.

12.2 Controllability of driftless models

In this section we consider ODE models in which the right side is *linear* with respect to the control variable, i.e. when (12.1) has the special form

$$\dot{x}(t) = g(x(t))u(t) = \sum_{k=1}^{m} g_k(x(t))u(t), \quad x(0) = \bar{x}_0, \tag{12.4}$$

where $g_k : X_0 \mapsto \mathbf{R}^n$ are given C^∞ (i.e. having continuous derivatives of arbitrary order) functions defined on an open subset X_0 of $\mathbf{R}^n$, and $u(t) = [u_1(t); \dots ; u_m(t)]$ is the vector control input. Note that linearization (12.2) of (12.4) around every equilibrium solution $x_0(t) \equiv \bar{x}_0 = \text{const}$, $u_0(t) = 0$ yields $A = 0$ and $B = g(\bar{x}_0)$, which means that the linearization is never controllable unless $m = n$. Nevertheless, it turns out that, for a "generic" function g, system (12.4) is expected to be completely controllable, as long as $m > 1$.

12.2.1 Local controllability and Lie brackets

Let us say that system (12.4) is *locally controllable* at a point $\bar{x}_0 \in X_0$ if for every $\epsilon > 0$, $T > 0$, and $\bar{x} \in X_0$ such that $|\bar{x} - \bar{x}_0| < \epsilon$ there exists a bounded measureable function $u : [0, T] \mapsto \mathbf{R}^m$ defining a solution of (12.4) with $x(0) = \bar{x}_0$ such that $x(T) = \bar{x}$ and

$$|x(t) - \bar{x}_0| < \epsilon \quad \forall\, t \in [0, T].$$

The local controlability conditions to be presented in this section are based on the notion of a *Lie bracket*. Let us write $h_3 = [h_1, h_2]$ (which reads as "h_3 is the Lie bracket of h_1 and h_2") when $h_k : X_0 \mapsto \mathbf{R}^n$ are continuous functions defined on an open subset X_0 of $\mathbf{R}^n$, functions h_1, h_2 are continuously differentiable on X_0, and

$$h_3(\bar{x}) = \dot{h}_1(\bar{x})h_2(\bar{x}) - \dot{h}_2(\bar{x})h_1(\bar{x})$$

for all $\bar{x} \in X_0$, where $\dot{h}_k(\bar{x})$ denotes the Jacobian of h_k at $\bar{x}$.

The reasoning behind the definition, as well as a more detailed study of the properties of Lie brackets, will be postponed until the proof of the controllability results of this subsection.

Let us call a set of functions $h_k : X_0 \mapsto \mathbf{R}^n$, $(k = 1, \dots, q)$ *complete* at a point $\bar{x} \in X_0$ if either the vectors $h_i(\bar{x})$ with $i = 1, \dots, m$ span the whole $\mathbf{R}^n$ or there exist functions $h_k : X_0 \mapsto \mathbf{R}^n$, $(k = q + 1, \dots, N)$, such that for every $k > q$ we have $h_k = [h_i, h_s]$ for some $i, s < k$, and the vectors $h_i(\bar{x})$ with $i = 1, \dots, N$ span the whole $\mathbf{R}^n$.

Theorem 12.3 *If C^∞ functions $g_k : X_0 \mapsto \mathbf{R}^n$ form a complete set at $\bar{x}_0 \in X_0$ then system (12.4) is locally controllable at $\bar{x}_0$.*

Theorem 12.3 provides a sufficient criterion of local controllability in terms of the span of all vector fields which can be generated by applying repeatedly the Lie bracket operation to g_k. This condition is not necessary, as can be seen from the following example: the second order system

$$\begin{aligned}\dot{x}_1 &= u_1,\\ \dot{x}_2 &= \phi(x_1)u_2,\end{aligned}$$

where function $\phi: \mathbf{R} \mapsto \mathbf{R}$ is infinitely many times continuously differentiable and such that

$$\phi(0) = 0, \ \phi(y) > 0 \text{ for } y \neq 0, \ \phi^{(k)}(0) = 0 \ \forall \ k,$$

is locally controllable at every point $\bar{x}_0 \in \mathbf{R}^n$ despite the fact that the corresponding set of vector fields

$$g_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ \phi(x_1) \end{bmatrix}$$

is not complete at $\bar{x} = 0$. On the other hand, the example of the system

$$\dot{x} = xu,$$

which is not locally controlable at $\bar{x} = 0$, but is defined by a (single element) set of vector fields which is complete at every point except $\bar{x} = 0$, shows that there is little room for relaxing the sufficient conditions of Theorem 12.3.

12.2.2 Proof of Theorem 12.3

Let $\mathcal{S}$ denote the set of all continuous functions $s: \ \Omega_s \mapsto X_0$, where Ω_s is an open subset of $\mathbf{R} \times X_0$ containing $\{0\} \times X_0$ (Ω_s is allowed to depend on s). Let $S_k \in \mathcal{S}$ be the elements of $\mathcal{S}$ defined by

$$S_k(\tau, \bar{x}) = x(\tau): \quad \dot{x}(t) = g_k(x(t)), \ x(0) = \bar{x}.$$

Let $\mathcal{S}_g$ be subset of $\mathcal{S}$ which consists of all functions which can be obtained by recursion

$$s_{k+1}(\bar{x}, \tau) = S_{\alpha(k)}(s_k(\bar{x}, \tau), \phi_k(\tau)), \quad \sigma_0(\bar{x}, \tau) = \bar{x},$$

where $\alpha(k) \in \{1, 2, \ldots, m\}$ and $\phi_k: \ \mathbf{R} \mapsto \mathbf{R}$ are continuous functions such that $\phi_k(0) = 0$. One can view elements of $\mathcal{S}_g$ as admissible state transitions in system (12.2) with piecewise constant control depending on parameter τ in such a way that $\tau = 0$ corresponds to the identity transition. Note that for every $s \in \mathcal{S}_g$ there exists an "inverse" $s' \in \mathcal{S}_g$ such that

$$s(s'(\bar{x}, \tau), \tau) = \bar{x} \ \ \forall \ (\bar{x}, \tau) \in \Omega_{s'},$$

defined by applying inverses $S_{\alpha(k)}(\cdot, -\phi_k(\tau))$ of the basic transformations $S_{\alpha(k)}(\cdot, \phi_k(\tau))$ in the reverse order.

Let us call a C^∞ function $h: \ X_0 \mapsto \mathbf{R}^n$ *implementable* in control system (12.4) if for every integer $k > 0$ there exists a function $s \in \mathcal{S}_g$ which is k times continuously differentiable in the region $\tau \geq 0$ and in the region $\tau \leq 0$, such that

$$s(\bar{x}, \tau) = \bar{x} + \tau h(\bar{x}) + o(\tau) \tag{12.5}$$

as $\tau \to 0, \tau \geq 0$ for all $\bar{x} \in X_0$. One can say that the value $h(\bar{x})$ of an implementable function $h(\cdot)$ at a given point $\bar{x} \in X_0$ describes a direction in which solutions of (12.4) can be steered from $\bar{x}$.

We will prove Theorem 12.3 by showing that Lie bracket of two implementable vector fields is also an implementable vector field. After this is done, an implicit function argument similar to one used in the proof of Theorem 12.2 shows local controllability of (12.4).

Now we need to prove two intermediate statements concerning the set of implementable vector fields. Remember that for the differential flow $(t, \bar{x}) \mapsto S^h(\bar{x}, t)$ defined by a smooth vector field h we have

$$S^h(S^h(\bar{x}, t_1, t_2) = S^h(\bar{x}, t_1 + t_2),$$

which, in particular, implies that

$$S^h(\bar{x}, t) = \bar{x} + th(\bar{x}) + \frac{t^2}{2}\dot{h}(\bar{x})h(\bar{x}) + O(t^3)$$

as $t \to 0$. This is not necessarily true for a general transition s from the definition of an implementable vector field h. However, the next Lemma shows that s can always be chosen to match the first k Taylor coefficients of S^h.

Lemma 12.1 *If h is implementable then for every integer $k > 0$ there exists a k times continuously differentiable function $s \in \mathcal{S}_g$ such that*

$$s(\bar{x}, \tau) = S^h(\bar{x}, \tau) + O(\tau^k). \tag{12.6}$$

Proof By assumption, (12.6) holds for $k = 2$ and $\tau \geq 0$, where s is N times continuously differentiable in the region $\tau \geq 0$, and N can be chosen arbitrarily large. Assume that for $\tau \geq 0$

$$s(\bar{x}, \tau) = S^h(\bar{x}, \tau) + \tau^k w(\bar{x}) + O(\tau^{k+1},$$

(which is implied by (12.6)), where $2 \leq k < N$ and w is continuously differentiable. Then

$$s'(\bar{x}, \tau) = S^h(\bar{x}, -\tau) - \tau^k w(\bar{x}) + O(\tau^{k+1},$$

and hence for every $a, b > 0$ the function

$$s_{a,b}(\bar{x}, \tau) = s(s'(s(\bar{x}, a\tau), b\tau), a\tau)$$

satisfies

$$s_{a,b}(\bar{x},\tau) = S^h(\bar{x},(2*a-b)\tau) + (2a^k - b^k)\tau^k w(\bar{x}) + O(\tau^{k+1}.$$

Since $k \geq 2$, one can choose a, b in such way that

$$2a - b = 1, \quad 2a^k = b^k,$$

which yields (12.6) with k increased by 1.

After (12.6) is established for $\tau \geq 0$, s can be defined for negative arguments by

$$s(\bar{x},-\tau) = s'(\bar{x},\tau), \quad \tau \geq 0,$$

which makes it $k-1$ times continuously differentiable. ■

Next lemma is a key result explaining the importance of Lie brackets in controllability analysis.

Lemma 12.2 *If vector fields h_1, h_2 are implementable then so is their Lie bracket $h = [h_2, h_1]$.*

Proof By Lemma 12.1, there exist $2*k+2$ times continuously differentiable (for $\tau \neq 0$) functions $s_1, s_2 \in \mathcal{S}_g$ such that

$$s_i(\bar{x},\tau) = \bar{x} + \tau h_i(\bar{x}) + \tau^2 \dot{h}_i(\bar{x}) h_i(\bar{x}) + o(\tau^2).$$

Hence (check this!), $s_3 \in \mathcal{S}_g$ defined by

$$s_3(\bar{x},\tau) = s_2(s_1(s_2(s_1(\bar{x},\tau),\tau),-\tau),-\tau),$$

satisfies

$$s_3(\bar{x},\tau) = \bar{x} + \tau^2 h(\bar{x}) + o(\tau^2).$$

Now for $i = 3, 4, \dots, 2*k+2$ let

$$s_{i+1}(\bar{x},\tau) = s_i(s_i(\bar{x},\tau/\sqrt{2}),-\tau/\sqrt{2}).$$

By induction,

$$s_{i+2}(\bar{x},\tau) = \bar{x} + \sum_{q=1}^{i} \tau^{2i} \beta_{iq}(\bar{x}) + o(\tau^{2i}),$$

i.e. the transformation from s_i to s_{i+1} removes the smallest odd power of τ in the Taylor expansion for s_i. Hence

$$s(\bar{x},\tau) = s_{2k+2}(\bar{x},\sqrt{\tau}), \quad \tau \geq 0$$

defines a k times continuously differentiable function for sufficiently small $\tau \geq 0$, and

$$s(\bar{x},\tau) = \bar{x} + \tau h(\bar{x}) + o(\tau)$$

for $\tau \geq 0$, $\tau \to 0$. ■

12.2.3 Frobenius Theorem

Let $g_k : \mathbf{R}^n \mapsto \mathbf{R}^n$, $k = 1, \dots, m$, be k times ($k \geq 1$) continuously differentiable functions. We will say that g_k *define a regular* C^k *distribution* $\mathcal{D}(\{g_k\})$ at a point $\bar{x}_0 \in \mathbf{R}^n$ if vectors $g_k(\bar{x}_0)$ are linearly independent. Let X_0 be an open subset of $\mathbf{R}^n$. The distribution $\mathcal{D}(\{g_k\})$ is called *involutive* on X_0 if the value $g_{ij}(\bar{x})$ of every Lie bracket $g_{ij} = [g_i, g_j]$ belongs to the linear span of $g_k(\bar{x})$ for every $\bar{x} \in X_0$. Finally, distribution $\mathcal{D}(\{g_k\})$ is called *completely* C^k *integrable* over X_0 if there exists a set a set of k times continuously differentiable functions $h_k : X_0 \mapsto \mathbf{R}$, $k = 1, \dots, n-m$, such that the gradients $\nabla h_k(\bar{x})$ are linearly independent for all $\bar{x} \in X_0$, and

$$\nabla h_i(\bar{x}) g_j(\bar{x}) = 0 \quad \forall\ \bar{x} \in X_0.$$

The following classical result gives a partial answer to the question of what happens to controllability when the Lie brackets of vector fields g_k do not span $\mathbf{R}^n$.

Theorem 12.4 *Let $\mathcal{D}(\{g_k\})$ define a C^r distribution ($r \geq 1$) which is regular at $\bar{x}_0 \in \mathbf{R}^n$. Then the following conditions are equivalent:*

(a) there exists an open set X_0 containing $\bar{x}_0$ such that $\mathcal{D}(\{g_k\})$ is completely C^r integrable over X_0;

(b) there exists an open set X_0 containing $\bar{x}_0$ such that $\mathcal{D}(\{g_k\})$ is involutive on X_0.

Essentially, the Frobenius theorem states that in the neigborhood of a point where the dimension of the vector fields generated by Lie brackets of a given driftless control system is maximal but still less than n, there exist non-constant functions of the state vector which remain constant along all solutions of the system equations.

The condition of regularity in Theorem 12.4 is essential. For example, when

$$n = 2,\ m = 1,\ \bar{x}_0 = 0 \in \mathbf{R}^2,\ g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix},$$

the distribution defined by g is smooth and involutive (because $[g, g] = 0$ for every vector field g), but not regular at $\bar{x}_0$. Consequently, the conclusion of Theorem 12.4 does not hold at $\bar{x}_0 = 0$, but is nevertheless valid in a neigborhood of all other points.

The "locality" of complete integrability is also essential for the theorem. For example, the vector field

$$g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 + (1 - x_2^2 - x_2^2)^2 \\ x_3 \\ -x_2 \end{bmatrix}$$

defines a smooth regular involutive distribution on the whole $\mathbf{R}^3$. However, the distribution is not completely integrable over $\mathbf{R}^3$, while it is still completely integrable in a neigborhood of every point.

12.2.4 Proof of Theorem 12.4

The implication (a)⇒(b) follows straightforwardly from the reachability properties of Lie brackets. Let us prove the implication (b)⇒(a).

Let S_k^τ denote the differential flow map associated with g_k, i.e. $S_k^\tau(\bar{x}) = x(\tau)$, where $x = x(t)$ is the solution of

$$\dot{x}(t) = g_k(x(t)), \quad x(0) = \bar{x}.$$

Let $\Delta(\bar{x})$ denote the span of $g_1(\bar{x}), \ldots, g_m(\bar{x})$. The following stetement, which relies on both regularity and involutivity of the family $\{g_k\}_{k=1}^m$, states that the Jacobian $D_k^t(\bar{x})$ of S_k^t at $\bar{x}$ maps $\Delta(\bar{x})$ onto $\Delta(S_k^t(\bar{x}))$. This a generalization of the (obvious) fact that, for a single vector field $g: \mathbf{R}^n \mapsto \mathbf{R}^n$, moving the initial condition $x(0)$ by $g(x(0))\delta$ of a solution $x = x(t)$ of $dx/dt = g(x)$ results in $x(t)$ shifted by $g(x(t))\delta + o(\delta)$.

Lemma 12.3 *Under the assumptions of Theorem 12.4,*

$$D_k^t(\bar{x})\Delta(\bar{x}) = \Delta(S_k^t(\bar{x})).$$

Proof According to the rules for differentiation with respect to initial conditions, for a fixed $\bar{x}$, $D_k(t) = D_k^t(\bar{x})$ satisfies the ODE

$$\frac{d}{dt}D_k(t) = \dot{g}_k(x(t))D_k(t), \quad D_k(0) = I,$$

where $x(t) = S_k^t(\bar{x})$, and $\dot{g}_k(\bar{x})$ denotes the Jacobian of g at $\bar{x}$. Hence

$$\bar{D}_k(t) = D_k(t)g(\bar{x}), \text{ where } g(\bar{x}) = [g_1(\bar{x}) \; g_2(\bar{x}) \; \ldots \; g_m(\bar{x})],$$

satisfies

$$\frac{d}{dt}\bar{D}_k(t) = \dot{g}_k(x(t))\bar{D}_k(t), \quad \bar{D}_k(0) = g(\bar{x}). \tag{12.7}$$

Note that the (12.7) is an ODE with a unique solution. Hence, it is sufficient to show that (12.7) has a solution of the form

$$\bar{D}_k(t) = g(x(t))\delta(t) = \sum_{i=1}^m g_i(x(t))\delta_k(t), \tag{12.8}$$

where $\delta = \delta(t)$ is a continuously differentiable m-by-m matrix valued function of time, and $\delta_i(t)$ is the i-th row of $\delta(t)$. Indeed, substituting into (12.7) yields

$$\sum_i [\dot{g}_i(x(t))g_k(x(t))\delta_k(t) + g_i(x(t))\dot{\delta}_k(t)] = \dot{g}_k(x(t))\sum_i g_i(x(t))\delta_k(t)$$

and $\delta(0) = I$. Equivalently,

$$g(x(t))\dot{\delta}(t) = A(t)\delta(t),$$

where $A(t)$ is the n-by-m matrix with columns $g_{ki}(x(t))$, $g_{ki} = [g_k, g_i]$. By involutivity and regularity, $A(t) = g(x(t))a(t)$ for some continuous m-by-m matrix valued function $a = a(t)$. Thus, the equation for $\delta(t)$ becomes

$$\dot{\delta}(t) = a(t)\delta(t), \quad \delta(0) = I,$$

hence existence of $\delta(t)$ such that $\bar{D}_k(t) = g(x(t))\delta(t)$ is guaranteed. ■

Let $g_{m+1}, \dots, g_n$ be C^∞ smooth functions $g_i : \mathbf{R}^n \mapsto \mathbf{R}^n$ such that vectors $g_1(\bar{x}_0), \dots, g_n(\bar{x}_0)$ form a basis in $\mathbf{R}^n$. (For example, the functions g_i with $i > m$ can be chosen *constant*). Consider the map

$$F(z) = S_1^{z_1}(S_2^{z_2}(\dots(S_n^{z_n}(\bar{x}_0))\dots)),$$

defined and k times continuously differentiable for $z = [z_1, \dots, z_n]$ in a neigborhood of zero in $\mathbf{R}^n$. F is a k times differentiable map defined in a neigborhood of $z = 0$, $\bar{x} = \bar{x}_0$. Since the Jacobian $\dot{F}(0)$ of F at zero, given by

$$\dot{F}(0) = [g_1(\bar{x}_0) \ g_2(\bar{x}_0) \ \dots \ g_n(\bar{x}_0)]$$

is not singular, by the implicit mapping theorem there exists a k times continuously differentiable function

$$z = H(x) = [h_n(x); h_{n-1}(x); \dots; h_1(x)]$$

defined in a neigborhood of $\bar{x}_0$, such that $F(H(x)) \equiv x$.

Let us show that functions $h_i = h_i(x)$ satisfy the requirements of Theorem 12.4. Indeed, differentiating the identity $F(H(x)) \equiv x$ yields

$$\dot{F}(H(x))\dot{H}(x) = I, \qquad (12.9)$$

where

$$\dot{H}(x) = [\nabla h_n(x); \nabla h_{n-1}(x); \dots; \nabla h_1(x)]$$

is the Jacobian of H at x, and

$$\dot{F}(z) = [f_1(z) \ f_2(z) \ \dots \ f_n(x)]$$

is the Jacobian of F at z. Hence vectors $f_i(z)$ form a basis, as well as the co-vectors $\nabla h_i(x)h$. By Lemma 12.3, vectors $f_i(z)$ with $i \le m$ belong to $\Delta(F(z)) = \Delta(x)$ (and hence, by linear independence, form a basis in $\Delta(x)$). On the other hand, (12.9) implies $\nabla h_r(x) f_i(H(x)) = 0$ for $r \le n - m$ and $i \le m$. Hence $\nabla h_r(x)$ for $r \le n - m$ are linearly independent and orthogonal to all $g_i(x)$ for $i \le m$.