6.231 DYNAMIC PROGRAMMING

LECTURE 21

LECTURE OUTLINE

- Review of approximate policy iteration
- Projected equation methods for policy evaluation
- Issues related to simulation-based implementation
- Multistep projected equation methods
- Bias-variance tradeoff
- Exploration-enhanced implementations
- Oscillations

REVIEW: PROJECTED BELLMAN EQUATION

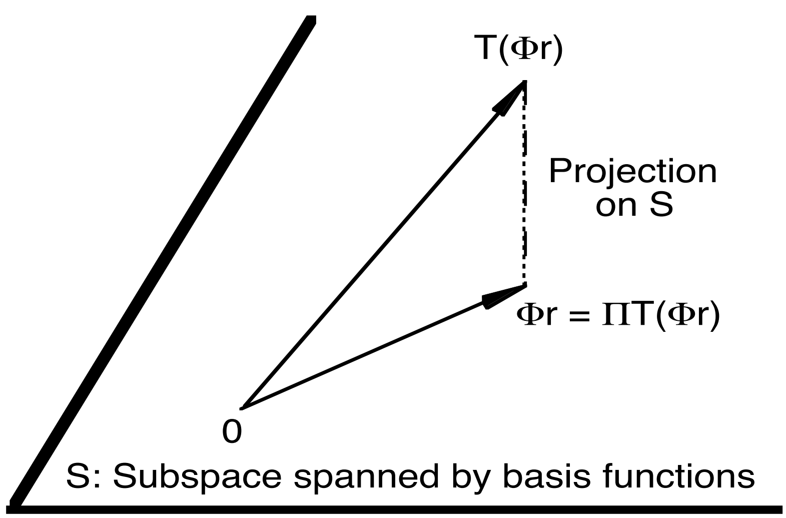
- For a fixed policy μ to be evaluated, consider the corresponding mapping T:

$$(TJ)(i) = \sum_{i=1}^{n} p_{ij}\big(g(i,j)+\alpha J(j)\big), \qquad i = 1,\ldots,n,$$

or more compactly, $TJ = g + \alpha PJ$

- Approximate Bellman's equation $J = TJ$ by $\Phi r = \Pi T(\Phi r)$ or the matrix form/orthogonality condition $Cr^* = d$, where

$$C = \Phi'\Xi(I - \alpha P)\Phi, \qquad d = \Phi'\Xi g.$$



Indirect method: Solving a projected form of Bellman's equation

PROJECTED EQUATION METHODS

- Matrix inversion: $r^* = C^{-1}d$
- Iterative Projected Value Iteration (PVI) method:

$$\Phi r_{k+1} = \Pi T(\Phi r_k) = \Pi(g + \alpha P \Phi r_k)$$

Converges to r^* if ΠT is a contraction. True if Π is projection w.r.t. steady-state distribution norm.

- Simulation-Based Implementations: Generate $k+1$ simulated transitions sequence $\{i_0, i_1, \ldots, i_k\}$ and approximations $C_k \approx C$ and $d_k \approx d$:

$$C_k = \frac{1}{k+1} \sum_{t=0}^{k} \phi(i_t)\big(\phi(i_t) - \alpha\phi(i_{t+1})\big)' \approx \Phi'\Xi(I - \alpha P)\Phi$$

$$d_k = \frac{1}{k+1} \sum_{t=0}^{k} \phi(i_t) g(i_t, i_{t+1}) \approx \Phi'\Xi g$$

- LSTD: $\hat{r}_k = C_k^{-1} d_k$
- LSPE: $r_{k+1} = r_k - G_k(C_k r_k - d_k)$ where

$$G_k \approx G = (\Phi'\Xi\Phi)^{-1}$$

Converges to r^* if ΠT is contraction.

ISSUES FOR PROJECTED EQUATIONS

- Implementation of simulation-based solution of projected equation $\Phi r \approx J_\mu$, where $C_k r = d_k$ and

$$C_k \approx \Phi' \Xi (I - \alpha P) \Phi, \qquad d_k \approx \Phi' \Xi g$$

- Low-dimensional linear algebra needed for the simulation-based approximations C_k and d_k (of order s; the number of basis functions).

- Very large number of samples needed to solve reliably nearly singular projected equations.

- Special methods for nearly singular equations by simulation exist; see Section 7.3 of the text.

- Optimistic (few sample) methods are more vulnerable to simulation error

- Norm mismatch/sampling distribution issue

- The problem of bias: Projected equation solution $\neq \Pi J_\mu$, the "closest" approximation of J_μ

- Everything said so far relates to policy evaluation. How about the effect of approximations on policy improvement?

- We will next address some of these issues

MULTISTEP METHODS

- Introduce a multistep version of Bellman's equation $J = T^{(\lambda)}J$, where for $\lambda \in [0,1)$,

$$T^{(\lambda)} = (1-\lambda)\sum_{\ell=0}^{\infty} \lambda^\ell T^{\ell+1}$$

Geometrically weighted sum of powers of T.

- T^ℓ is a contraction with mod. α^ℓ, w. r. to weighted Euclidean norm $\|\cdot\|_\xi$, where ξ is the steady-state probability vector of the Markov chain.

- Hence $T^{(\lambda)}$ is a contraction with modulus

$$\alpha_\lambda = (1-\lambda)\sum_{\ell=0}^{\infty} \alpha^{\ell+1}\lambda^\ell = \frac{\alpha(1-\lambda)}{1-\alpha\lambda}$$

Note $\alpha_\lambda \to 0$ as $\lambda \to 1$ - affects norm mismatch

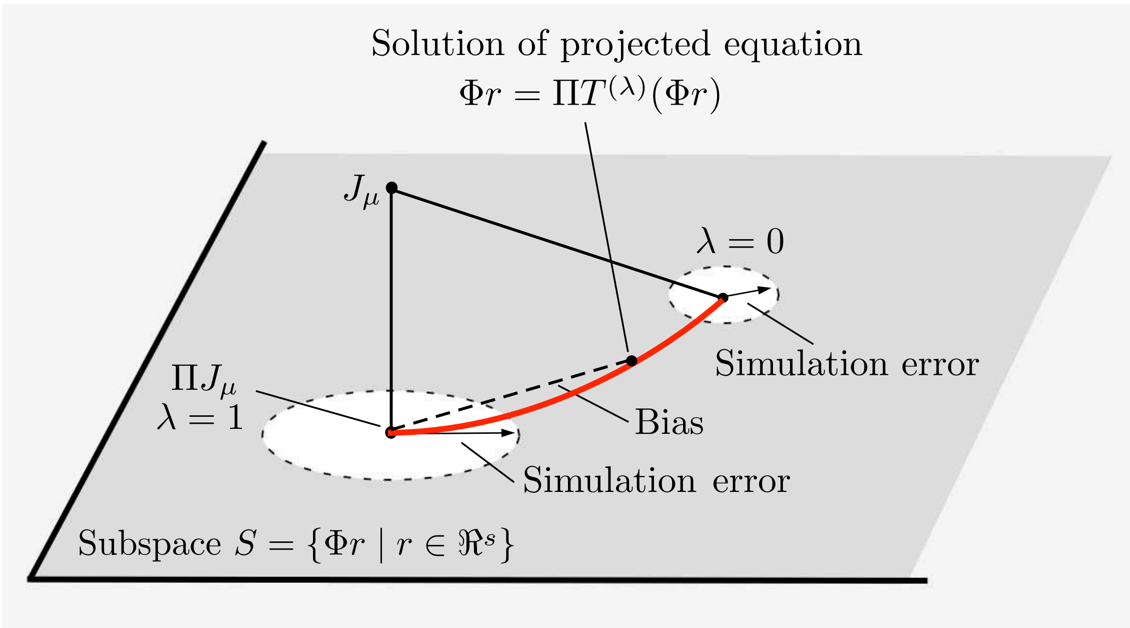
- T^ℓ and $T^{(\lambda)}$ have the same fixed point J_μ and

$$\|J_\mu - \Phi r_\lambda^*\|_\xi \le \frac{1}{\sqrt{1-\alpha_\lambda^2}}\,\|J_\mu - \Pi J_\mu\|_\xi$$

where Φr_λ^* is the fixed point of $\Pi T^{(\lambda)}$.

- Φr_λ^* depends on λ.

BIAS-VARIANCE TRADEOFF



- From $\|J_\mu - \Phi r_{\lambda,\mu}\|_\xi \leq \frac{1}{\sqrt{1-\alpha_\lambda^2}} \|J_\mu - \Pi J_\mu\|_\xi$

error bound

- As $\lambda \uparrow 1$, we have $\alpha_\lambda \downarrow 0$, so error bound (and quality of approximation) improves:

$$\lim_{\lambda \uparrow 1} \Phi r_{\lambda,\mu} = \Pi J_\mu$$

- But the simulation noise in approximating

$$T^{(\lambda)} = (1-\lambda) \sum_{\ell=0}^{\infty} \lambda^\ell T^{\ell+1}$$

increases

- Choice of λ is usually based on trial and error

MULTISTEP PROJECTED EQ. METHODS

- The multistep projected Bellman equation is

$$\Phi r = \Pi T^{(\lambda)}(\Phi r)$$

- In matrix form: $C^{(\lambda)} r = d^{(\lambda)}$, where

$$C^{(\lambda)} = \Phi' \Xi \big(I - \alpha P^{(\lambda)} \big) \Phi, \qquad d^{(\lambda)} = \Phi' \Xi g^{(\lambda)},$$

with

$$P^{(\lambda)} = (1 - \lambda) \sum_{\ell=0}^{\infty} \alpha^\ell \lambda^\ell P^{\ell+1}, \quad g^{(\lambda)} = \sum_{\ell=0}^{\infty} \alpha^\ell \lambda^\ell P^\ell g$$

- The LSTD(λ) method is $\big(C_k^{(\lambda)}\big)^{-1} d_k^{(\lambda)}$, where $C_k^{(\lambda)}$ and $d_k^{(\lambda)}$ are simulation-based approximations of $C^{(\lambda)}$ and $d^{(\lambda)}$.

- The LSPE(λ) method is

$$r_{k+1} = r_k - \gamma G_k \big(C_k^{(\lambda)} r_k - d_k^{(\lambda)} \big)$$

where G_k is a simulation-based approx. to $(\Phi' \Xi \Phi)^{-1}$

- TD(λ): An important simpler/slower iteration [similar to LSPE(λ) with $G_k = I$ - see the text].

MORE ON MULTISTEP METHODS

- The simulation process to obtain $C_k^{(\lambda)}$ and $d_k^{(\lambda)}$ is similar to the case $\lambda = 0$ (single simulation trajectory $i_0, i_1, \ldots$, more complex formulas)

$$C_k^{(\lambda)} = \frac{1}{k+1} \sum_{t=0}^{k} \phi(i_t) \sum_{m=t}^{k} \alpha^{m-t} \lambda^{m-t} \big(\phi(i_m) - \alpha \phi(i_{m+1})\big)'$$

$$d_k^{(\lambda)} = \frac{1}{k+1} \sum_{t=0}^{k} \phi(i_t) \sum_{m=t}^{k} \alpha^{m-t} \lambda^{m-t} g_{i_m}$$

- In the context of approximate policy iteration, we can use optimistic versions (few samples between policy updates).
- Many different versions (see the text).
- Note the λ-tradeoffs:
  - As $\lambda \uparrow 1$, $C_k^{(\lambda)}$ and $d_k^{(\lambda)}$ contain more "simulation noise", so more samples are needed for a close approximation of $r_{\lambda,\mu}$
  - The error bound $\|J_\mu - \Phi r_{\lambda,\mu}\|_\xi$ becomes smaller
  - As $\lambda \uparrow 1$, $\Pi T^{(\lambda)}$ becomes a contraction for arbitrary projection norm

APPROXIMATE PI ISSUES - EXPLORATION

- 1st major issue: exploration. Common remedy is the off-policy approach: Replace P of current policy with

$$\overline{P} = (I - B)P + BQ,$$

where B is a diagonal matrix with $\beta_i \in [0, 1]$ on the diagonal, and Q is another transition matrix.

- Then LSTD and LSPE formulas must be modified ... otherwise the policy associated with $\overline{P}$ (not P) is evaluated (see the textbook, Section 6.4).

- Alternatives: Geometric and free-form sampling

- Both of these use multiple short simulated trajectories, with random restart state, chosen to enhance exploration (see the text)

- Geometric sampling uses trajectories with geometrically distributed number of transitions with parameter $\lambda \in [0, 1)$. It implements LSTD(λ) and LSPE(λ) with exploration.

- Free-form sampling uses trajectories with more generally distributed number of transitions. It implements method for approximation of the solution of a generalized multistep Bellman equation.

APPROXIMATE PI ISSUES - OSCILLATIONS

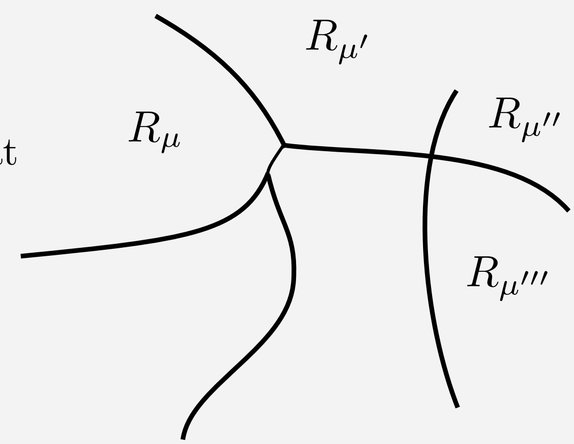
- Define for each policy μ

$$R_\mu = \big\{ r \mid T_\mu(\Phi r) = T(\Phi r) \big\}$$

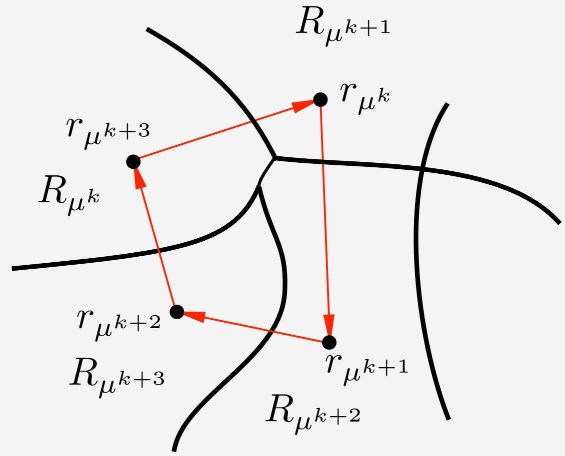
- These sets form the greedy partition of the parameter r-space

$$R_\mu = \big\{ r \mid T_\mu(\Phi r) = T(\Phi r) \big\}$$

For a policy μ, R_μ is the set of all r such that policy improvement based on Φr produces μ

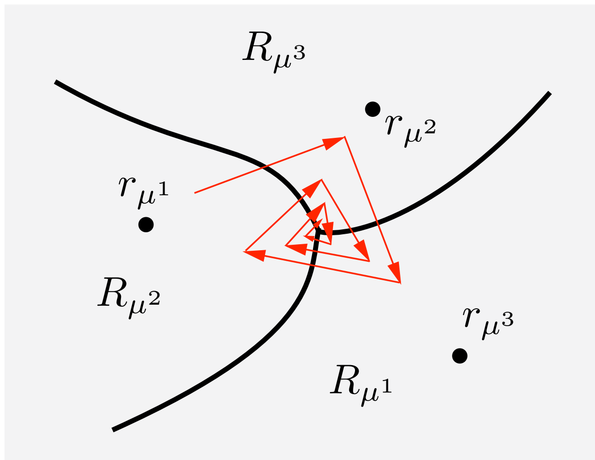


- Oscillations of nonoptimistic approx.: r_μ is generated by an evaluation method so that $\Phi r_\mu \approx J_\mu$



MORE ON OSCILLATIONS/CHATTERING

- For optimistic PI a different picture holds



- Oscillations are less violent, but the "limit" point is meaningless!

- Fundamentally, oscillations are due to the lack of monotonicity of the projection operator, i.e., $J \le J'$ does not imply $\Pi J \le \Pi J'$.

- If approximate PI uses policy evaluation

$$\Phi r = (WT_\mu)(\Phi r)$$

with W a monotone operator, the generated policies converge (to an approximately optimal limit).

- The operator W used in the aggregation approach has this monotonicity property.