6.189 IAP 2007

Lecture 15

Cilk

Design and Analysis of Dynamic Multithreaded Algorithms

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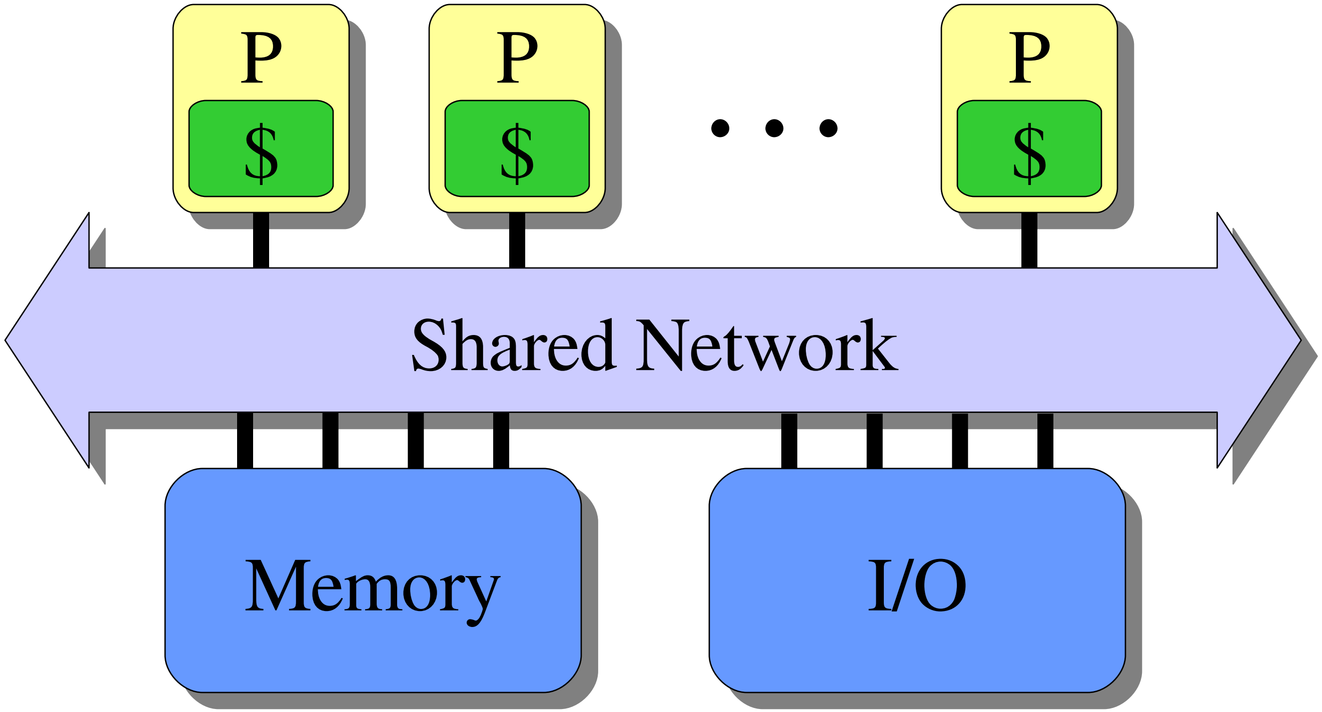
Shared-Memory Multiprocessor



- ***Symmetric multiprocessor (SMP)***
- ***Cache-coherent nonuniform memory architecture (CC-NUMA)***

Cilk

A C language for dynamic multithreading with a provably good runtime system.

Platforms

- Sun UltraSPARC Enterprise
- SGI Origin 2000
- Compaq/Digital Alphaserver
- Intel Pentium SMP's

Applications

- virus shell assembly
- graphics rendering
- *n*-body simulation
- ★Socrates and Cilkchess

Cilk automatically manages low-level aspects of parallel execution, including protocols, load balancing, and scheduling.

Fibonacci

```
int fib (int n)
if (n<2) return (n);
  else
    int x,y;
    x = fib(n-1);
    y = fib(n-2);
    return (x+y);
  }
}
```

C elision

Cilk code

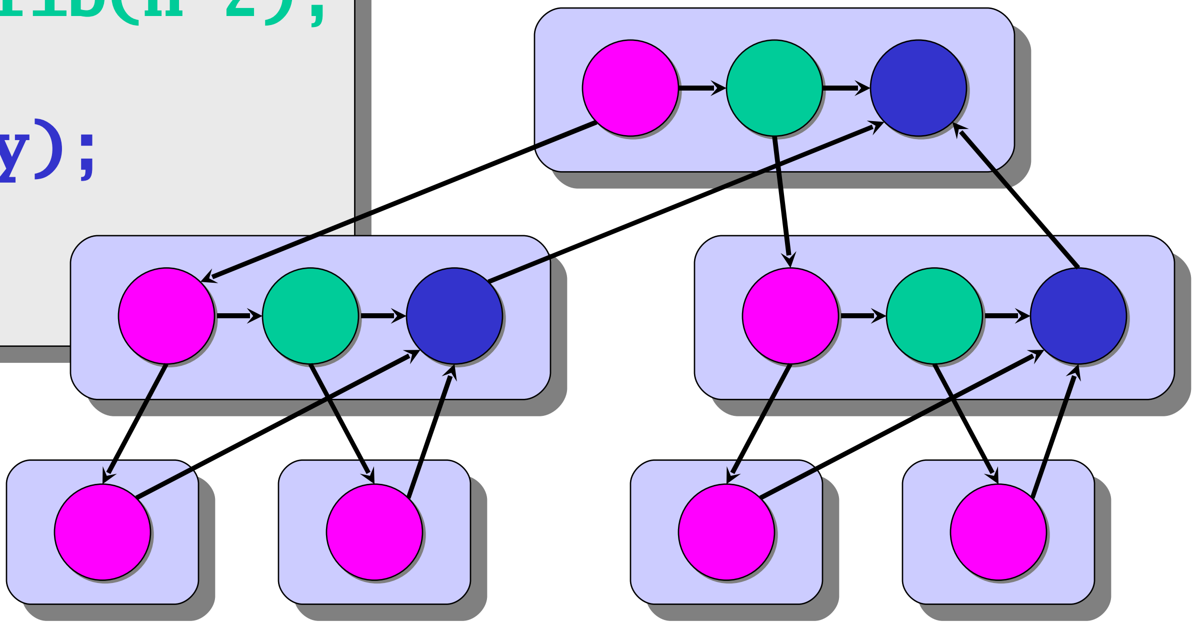
```
cilk int fib (int n)
  if (n<2) return (n);
  else
    int x,y;
    x = spawn fib(n-1);
    y = spawn fib(n-2);
    sync;
    return (x+y);
  }
}
```

Cilk is a ***faithful*** extension of C. A Cilk program's ***serial elision*** is always a legal implementation of Cilk semantics. Cilk provides ***no*** new data types.

Dynamic Multithreading

```
cilk int fib (int n) {
  if (n<2) return (n);
  else {
    int x,y;
    x = spawn fib(n-1);
    y = spawn fib(n-2);
    sync;
    return (x+y);
  }
}
```

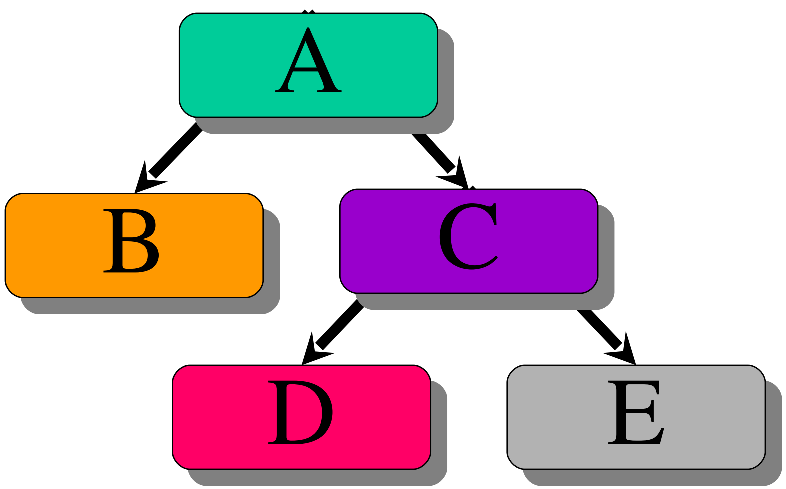
*The **computation dag** unfolds dynamically.*

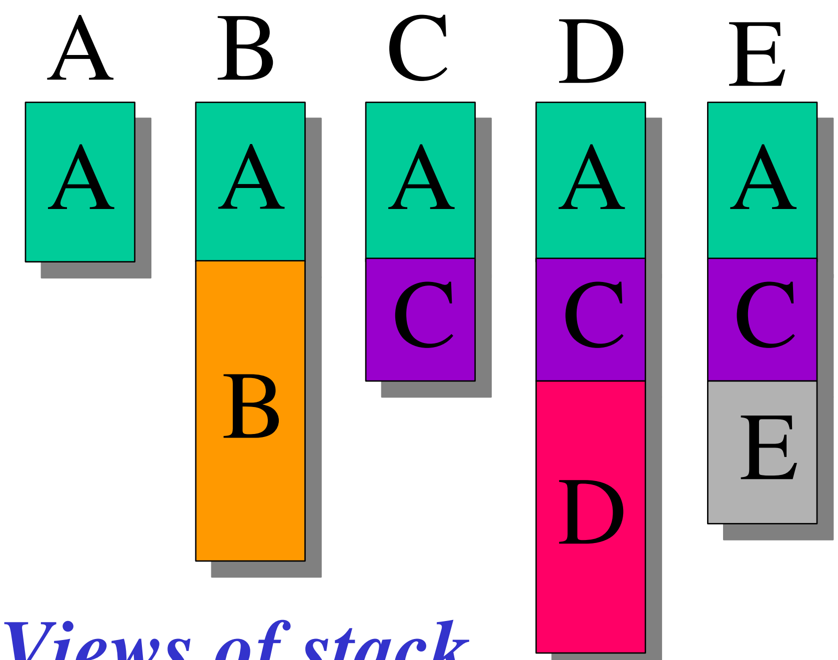


"Processor oblivious."

Cactus Stack

Cilk supports C's rule for pointers: A pointer to stack space can be passed from parent to child, but not from child to parent. (Cilk also supports **`malloc`**.)





Views of stack

Cilk's ***cactus stack*** supports several views in parallel.

Advanced Features

- Returned values can be incorporated into the parent frame using a delayed internal function called an ***inlet***:

```
int y;
inlet void foo (int x) {
  if (x > y) y = x;
}
…
spawn foo(bar(z));
```

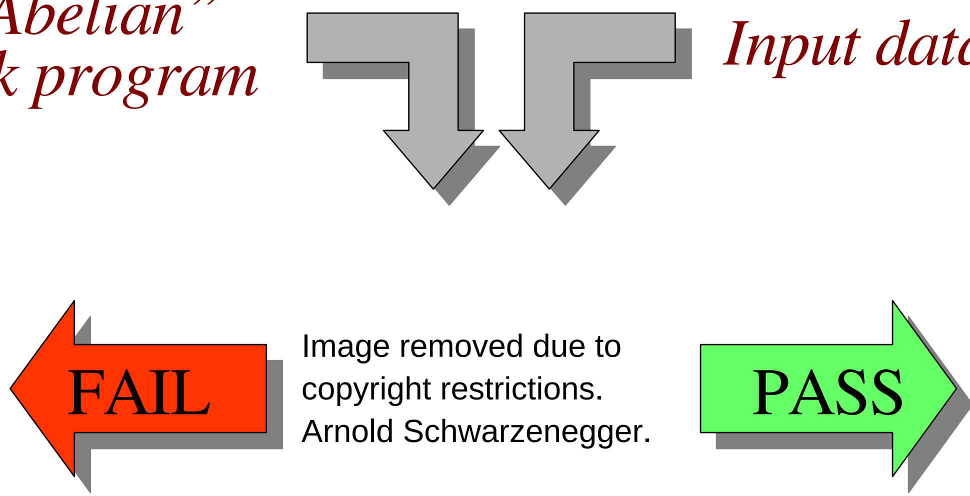
- Within an inlet, the **`abort`** keyword causes all other children of the parent frame to be terminated.
- The **SYNCHED** pseudovariable tests whether a **`sync`** would succeed.
- A Cilk library provides ***mutex locks*** for atomicity.

Debugging Support

The ***Nondeterminator*** debugging tool detects and localizes data-race bugs.

"Abelian" Cilk program

Input data set



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Information localizing a data race.

Every scheduling produces the same result.

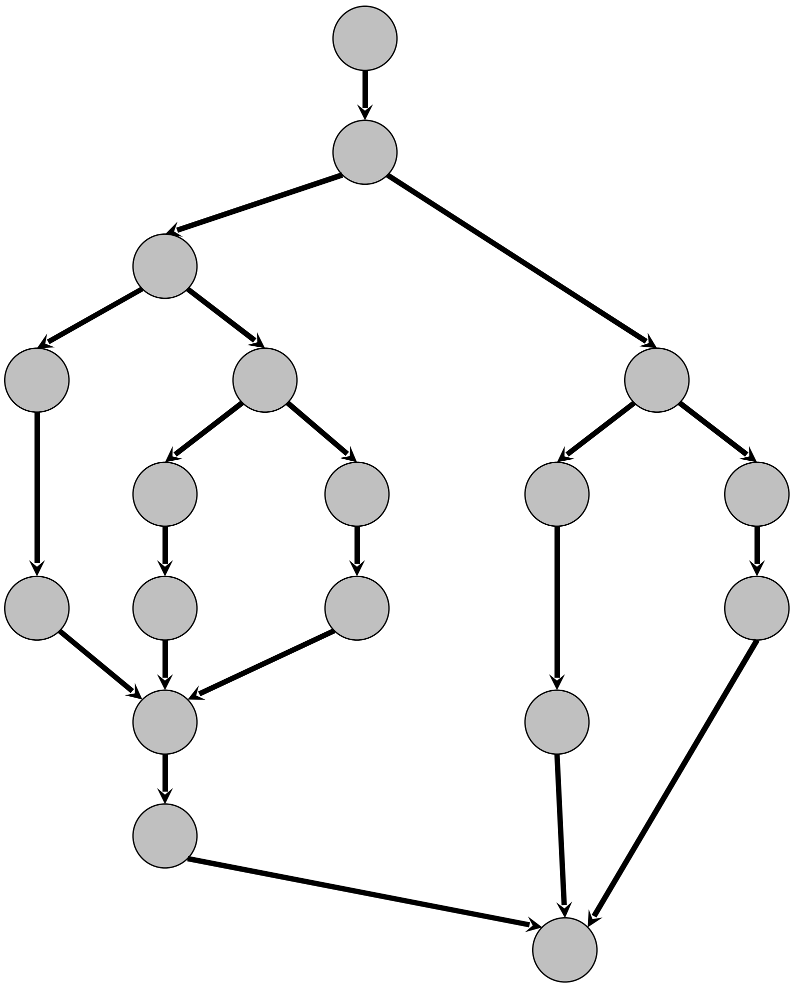
A ***data race*** occurs whenever a thread modifies location and another thread, holding no locks in common, accesses the location simultaneously.

Outline

- Theory and Practice
- A Chess Lesson
- Fun with Algorithms
- Work Stealing
- Opinion & Conclusion

Algorithmic Complexity Measures

T_P = execution time on P processors



Algorithmic Complexity Measures

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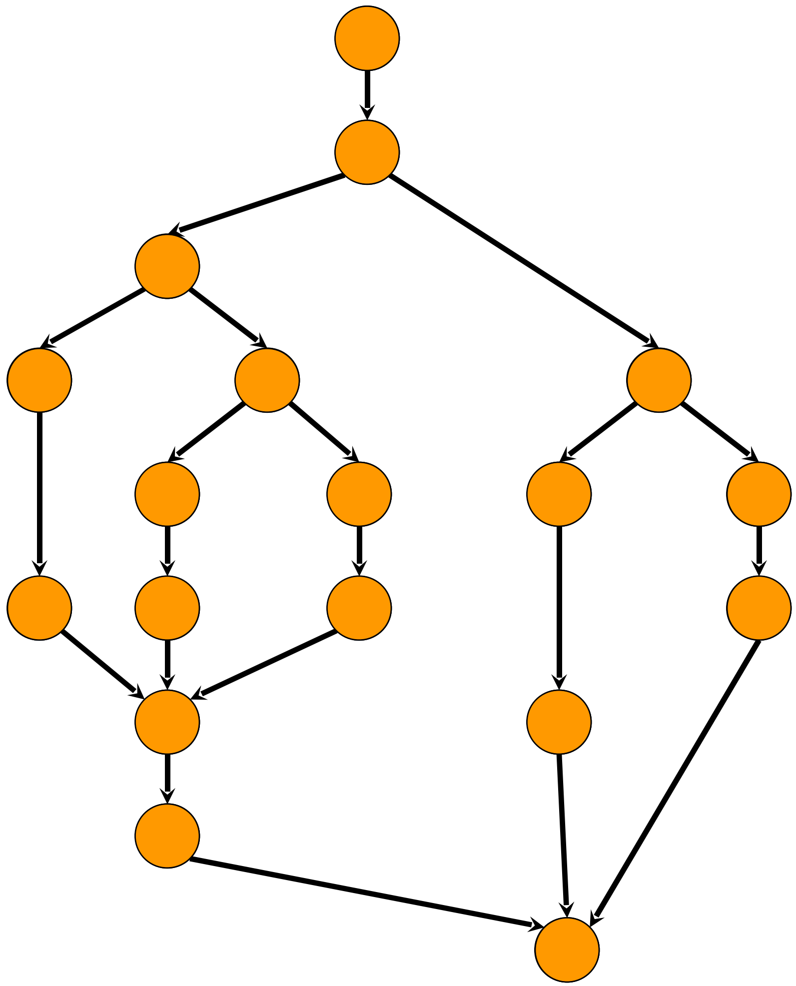

T_1 = ***work***

Algorithmic Complexity Measures

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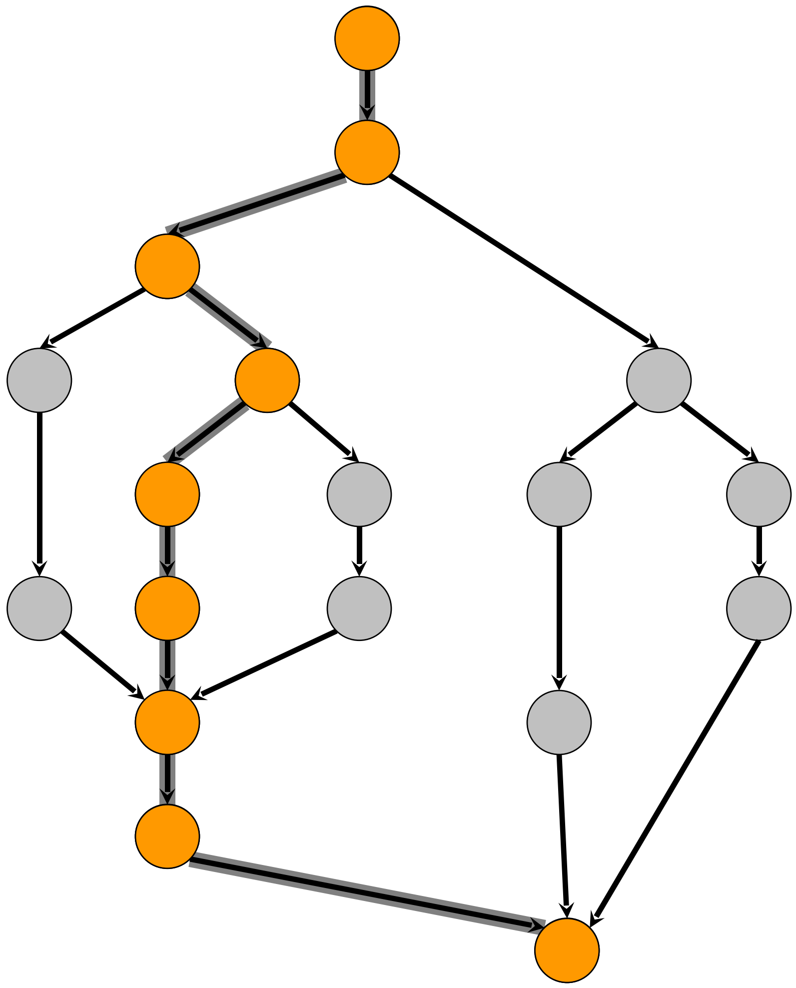

T_1 = ***work***

T_∞ = ***critical path***

Algorithmic Complexity Measures

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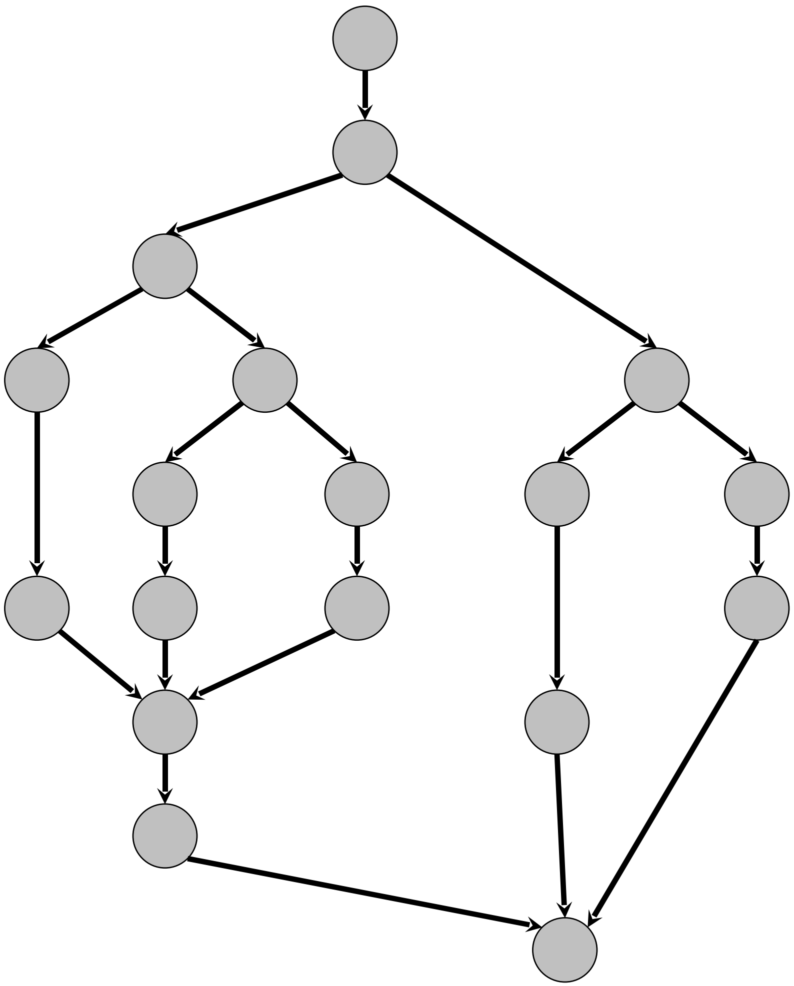

T_1 = ***work***

T_∞ = ***critical path***

Lower Bounds

- $T_P \geq T_1/P$
- $T_P \geq T_\infty$

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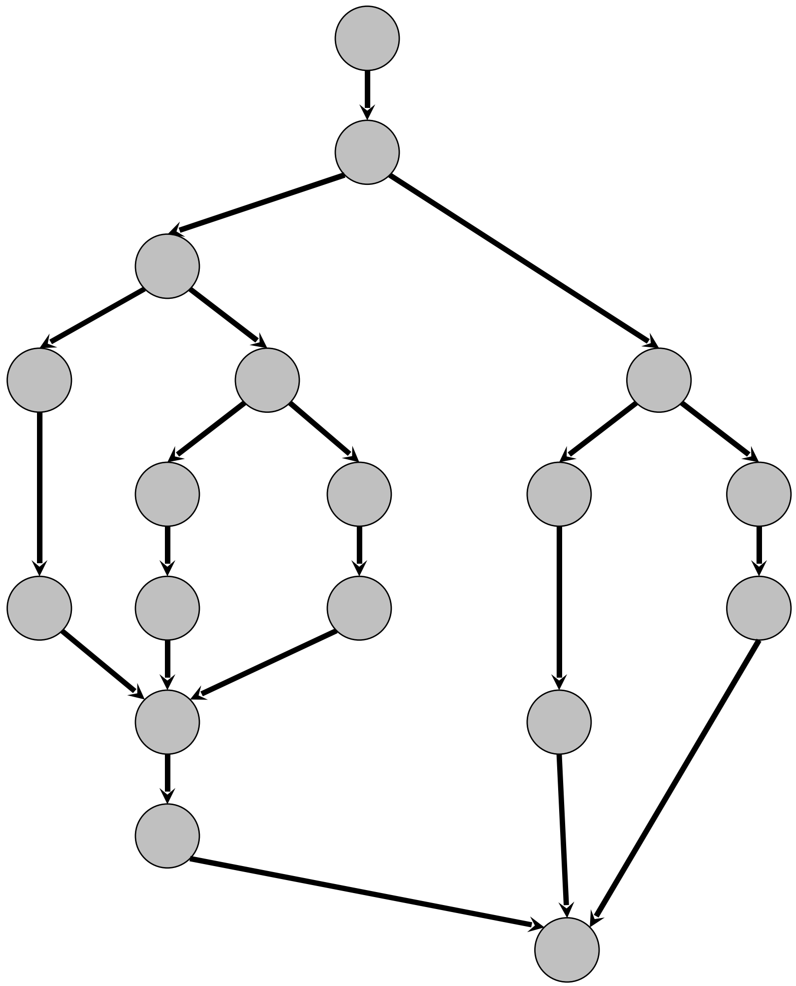


T_1 = ***work***

T_∞ = ***critical path***

Lower Bounds

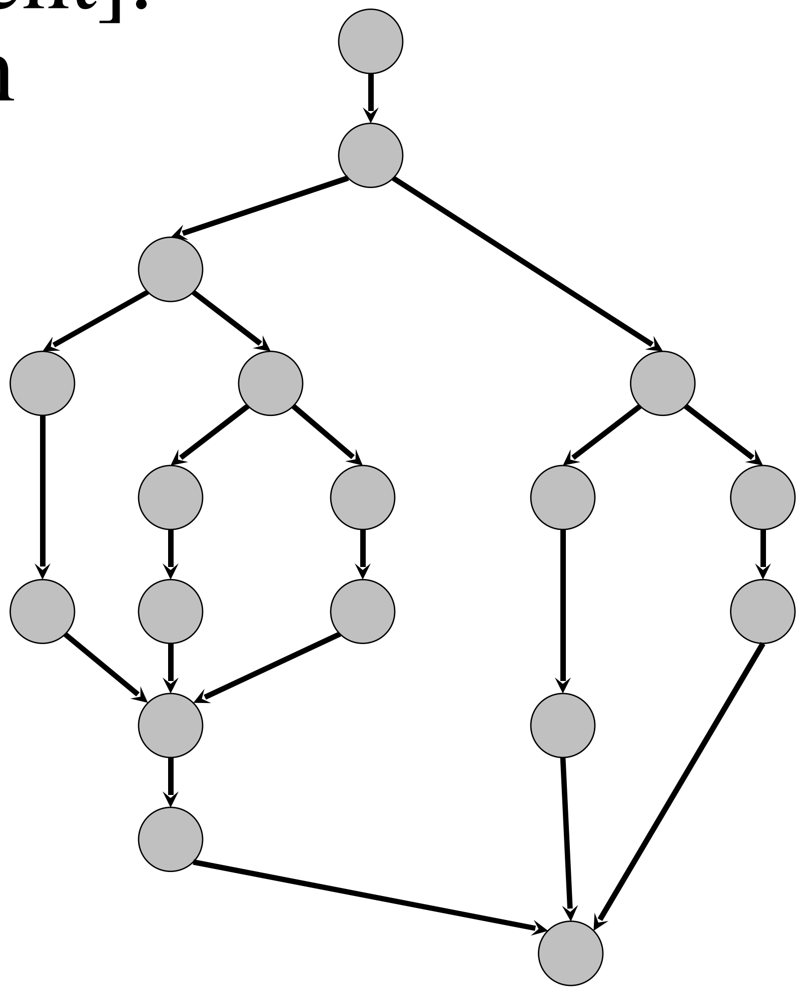
- $T_P \geq T_1/P$
- $T_P \geq T_\infty$

T_1/T_P = ***speedup***

T_1/T_∞ = ***parallelism***

Greedy Scheduling

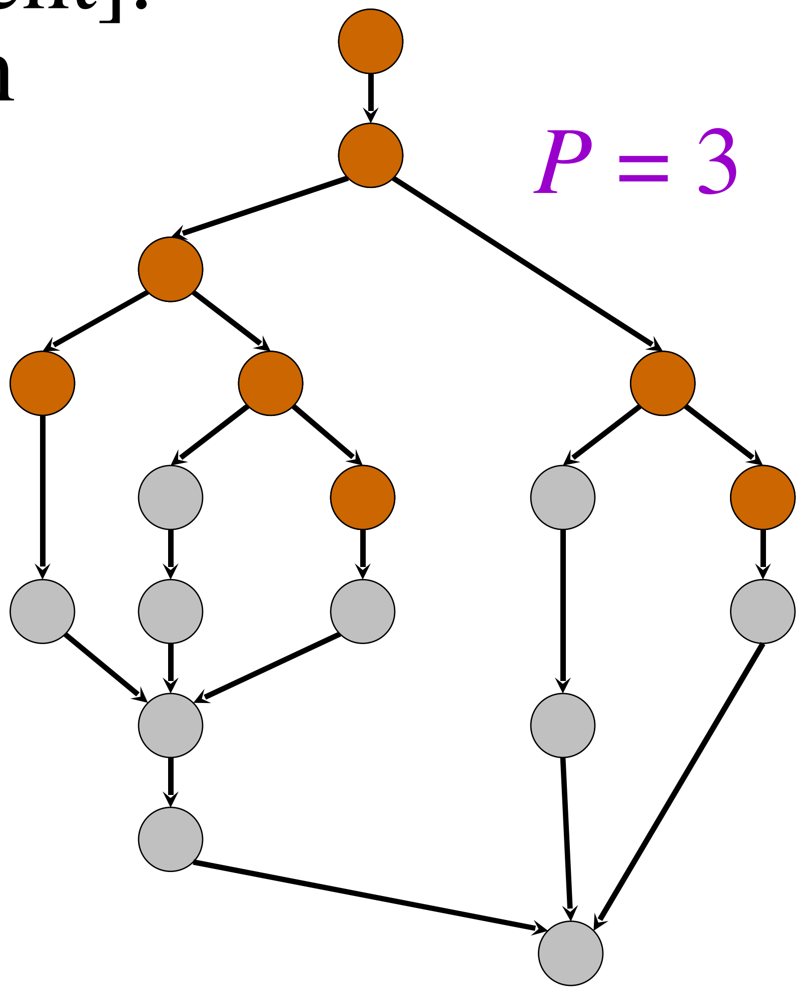
Theorem [Graham & Brent]: There exists an execution with $T_P \leq T_1/P + T_\infty$.



Greedy Scheduling

Theorem [Graham & Brent]:
There exists an execution with $T_P \leq T_1/P + T_\infty$.

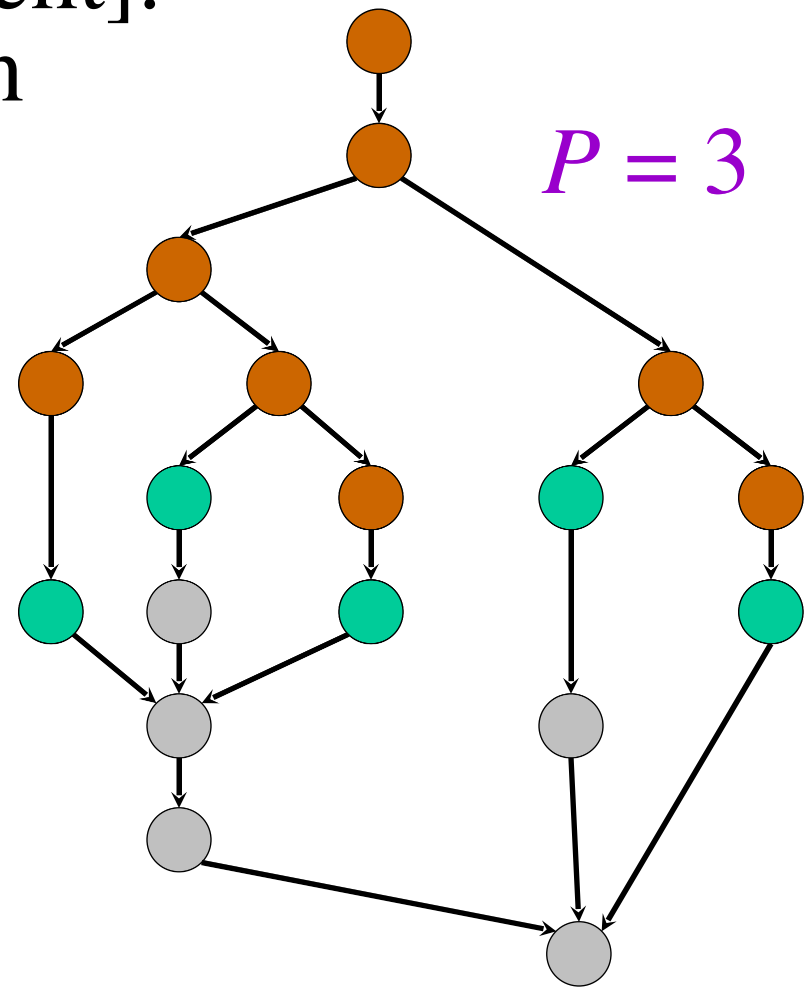
Proof. At each time step, …



Greedy Scheduling

Theorem [Graham & Brent]: There exists an execution with $T_P \leq T_1/P + T_\infty$.

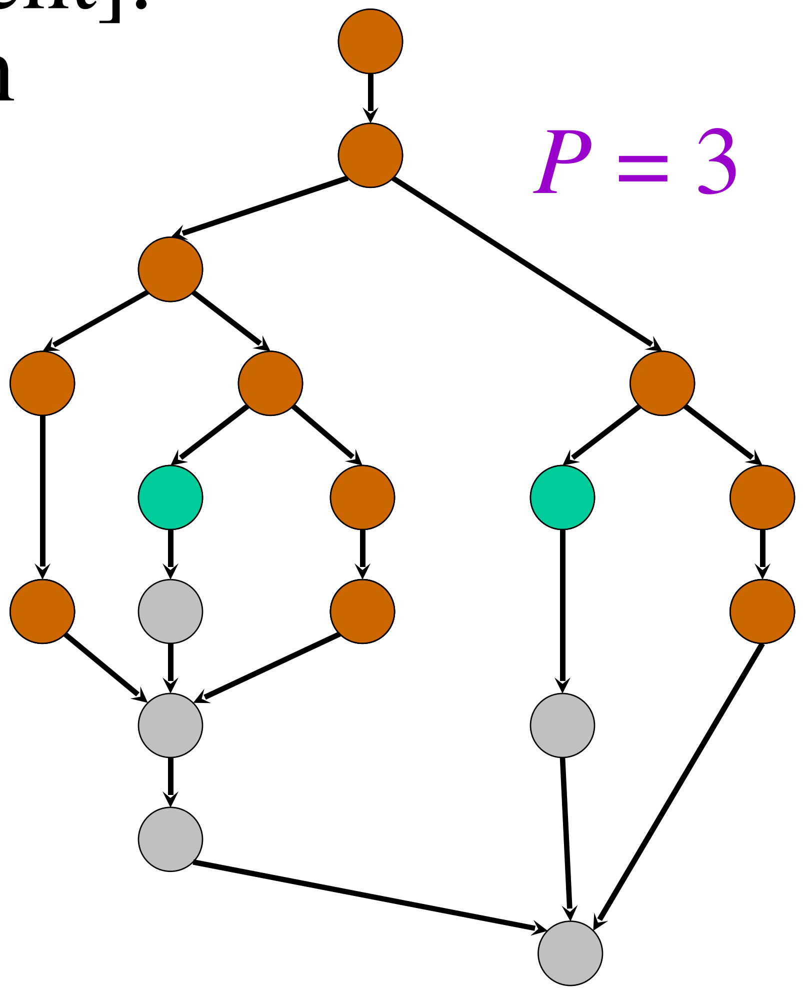
Proof. At each time step, if at least P tasks are ready, …



Greedy Scheduling

Theorem [Graham & Brent]: There exists an execution with $T_P \leq T_1/P + T_\infty$.

Proof. At each time step, if at least P tasks are ready, execute P of them.



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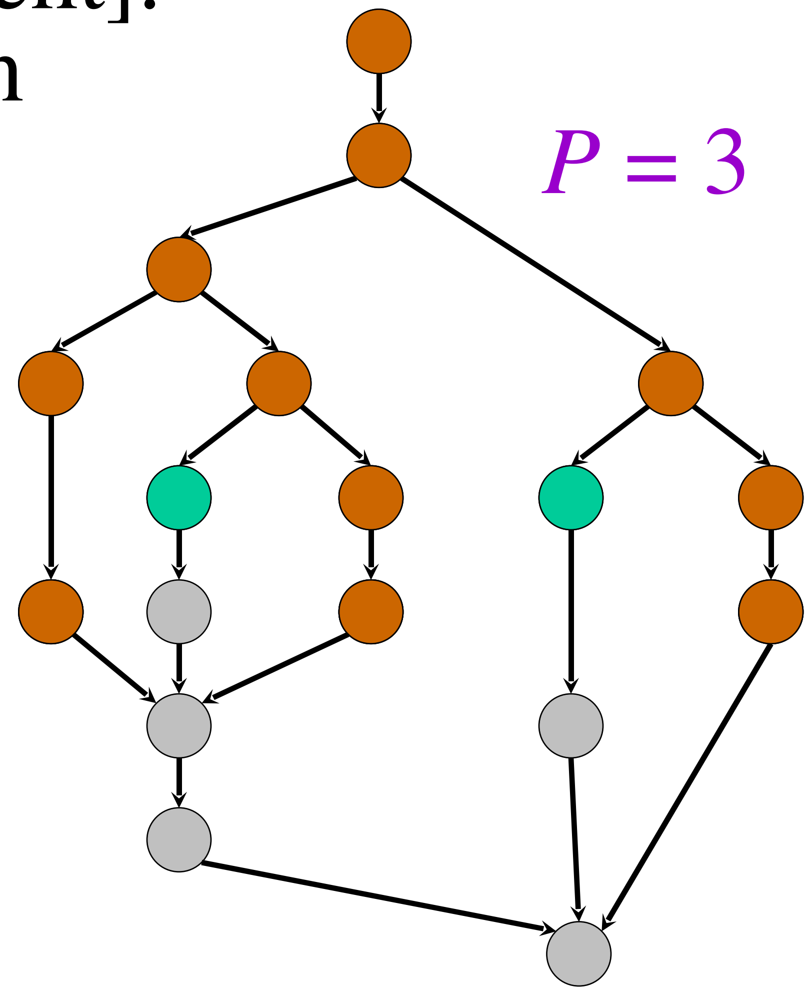

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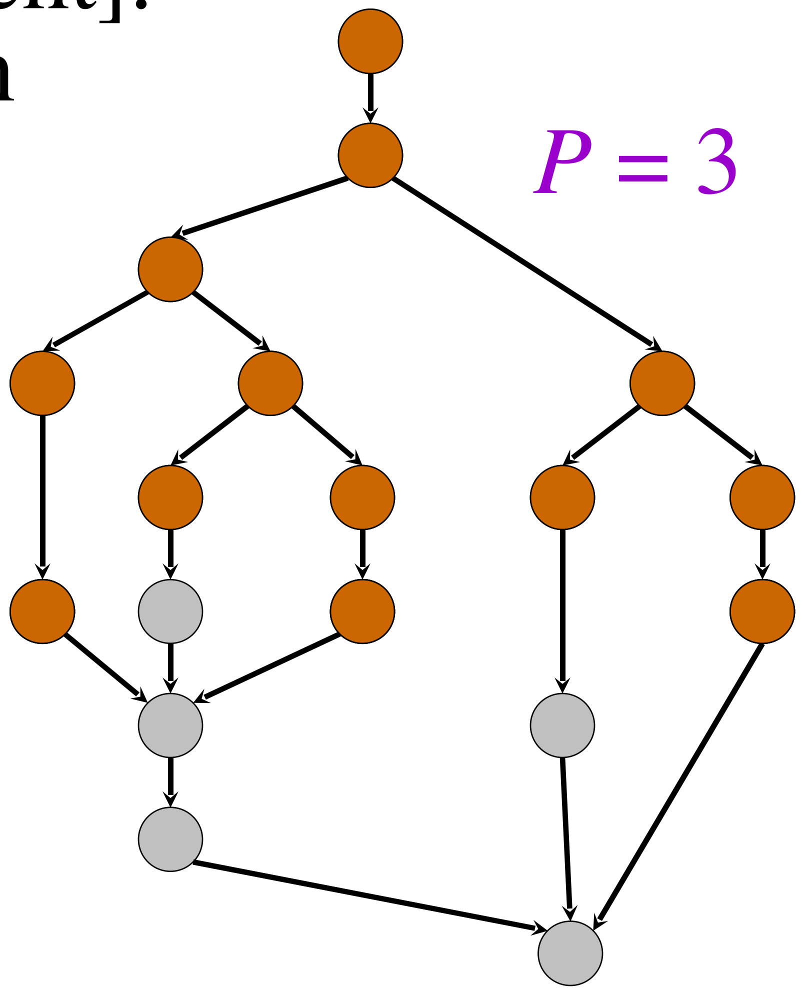

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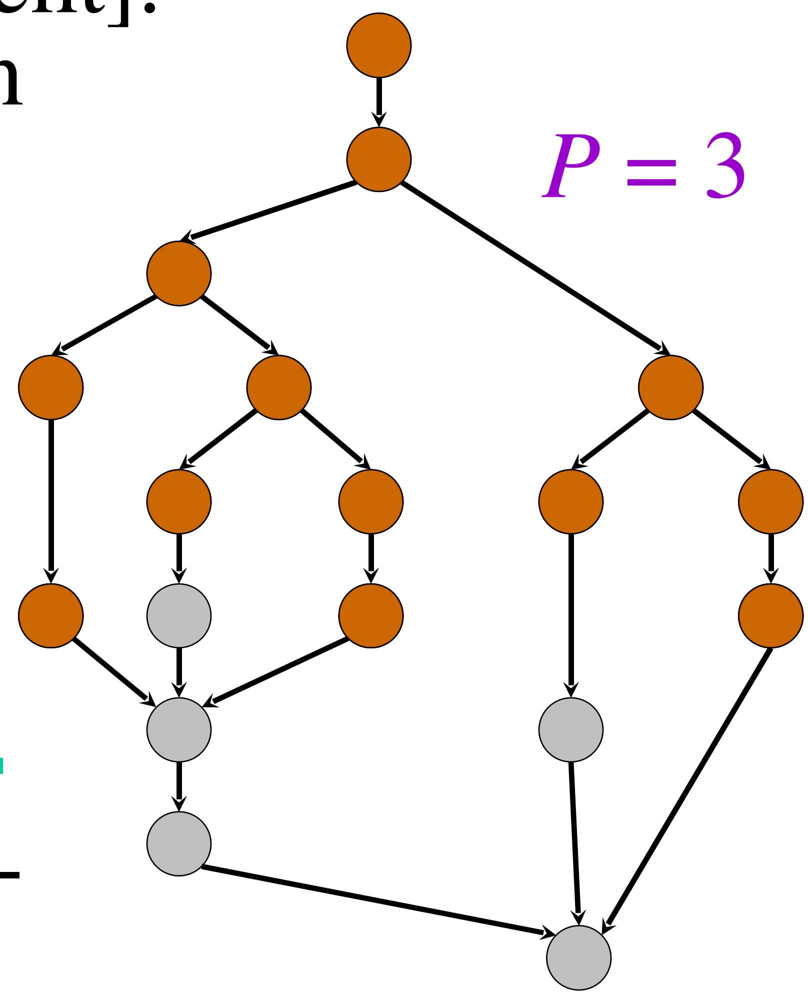

Corollary: Linear speed-up when $P \le T_1/T_\infty$.

Cilk Performance

Cilk's ***"work-stealing"*** scheduler achieves

- $T_P = T_1/P + O(T_\infty)$ expected time (provably);
- $T_P \approx T_1/P + T_\infty$ time (empirically).

Near-perfect linear speedup if $P \leq T_1/T_\infty$.

Instrumentation in Cilk provides accurate measures of T_1 and T_∞ to the user.

The average cost of a **spawn** in Cilk-5 is only 2–6 times the cost of an ordinary C function call, depending on the platform.

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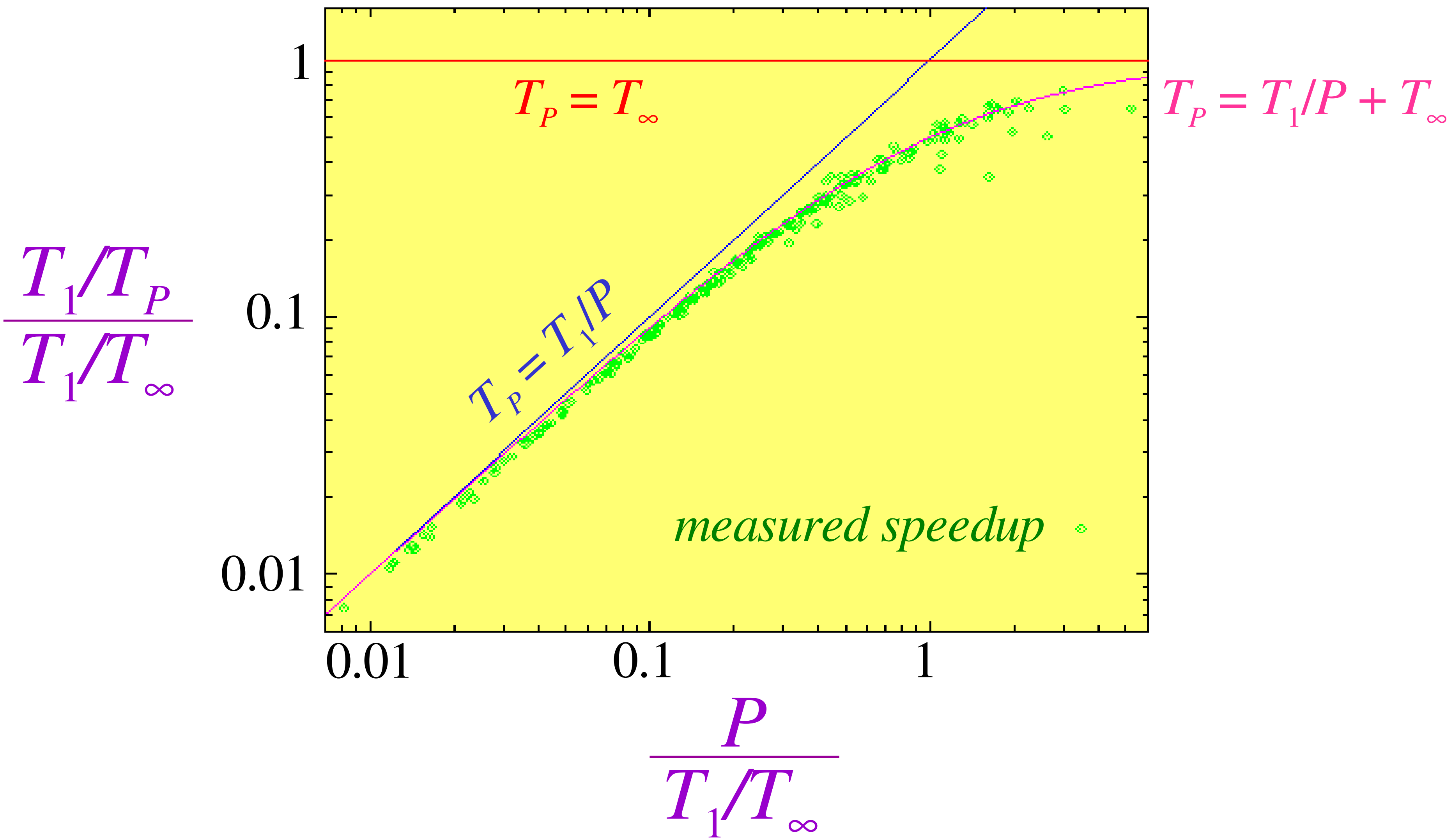
Cilk Chess Programs

Socrates placed 3rd in the 1994 International Computer Chess Championship running on NCSA's 512-node Connection Machine CM5.

Socrates 2.0 took 2nd place in the 1995 World Computer Chess Championship running on Sandia National Labs' 1824-node Intel Paragon.

- *Cilkchess* placed 1st in the 1996 Dutch Open running on a 12-processor Sun Enterprise 5000. It placed 2nd in 1997 and 1998 running on Boston University's 64-processor SGI Origin 2000.
- *Cilkchess* tied for 3rd in the 1999 WCCC running on NASA's 256-node SGI Origin 2000.

Socrates Normalized Speedup



Socrates Speedup Paradox

Original program

$T_{32} = 65$ seconds

Proposed program

$T'_{32} = 40$ seconds

$$T_P \approx T_1/P + T_\infty$$

Original program:

$T_1 = 2048$ seconds

$T_\infty = 1$ second

$T_{32} = 2048/32 + 1$
$= 65$ seconds

$T_{512} = 2048/512 + 1$
$= 5$ seconds

Proposed program:

$T'_1 = 1024$ seconds

$T'_\infty = 8$ seconds

$T'_{32} = 1024/32 + 8$
$= 40$ seconds

$T'_{512} = 1024/512 + 8$
$= 10$ seconds

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Matrix Multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

C A B

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Recursive Matrix Multiplication

Divide and conquer on $n \times n$ matrices.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix} + \begin{bmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

8 multiplications of $(n/2) \times (n/2)$ matrices.
1 addition of $n \times n$ matrices.

Matrix Multiplication in Cilk

```
cilk Mult(*C,*A,*B,n)
{ float T[n][n];
  h base case & partition matrices i
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2);
  spawn Mult(C22,A21,B12,n/2);
  spawn Mult(C21,A21,B11,n/2);
  spawn Mult(T11,A12,B21,n/2);
  spawn Mult(T12,A12,B22,n/2);
  spawn Mult(T22,A22,B22,n/2);
  spawn Mult(T21,A22,B21,n/2);
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

$C = AB$

(Coarsen base cases for efficiency.)

$C = C + T$

```
cilk Add(*C,*T,n)
{    h base case & partition matrices i
     spawn Add(C11,T11,n/2);
     spawn Add(C12,T12,n/2);
     spawn Add(C21,T21,n/2);
     spawn Add(C22,T22,n/2);
     sync;
     return;
}
```

Analysis of Matrix Addition

```
cilk Add(*C,*T,n)
{    h base case & partition matrices i
     spawn Add(C11,T11,n/2);
     spawn Add(C12,T12,n/2);
     spawn Add(C21,T21,n/2);
     spawn Add(C22,T22,n/2);
     sync;
     return;
}
```

Work: $A_1(n) = 4\,A_1(n/2) + (1)$

$= (n^2)$

Critical path: $A_\infty(n) = A_\infty(n/2) + (1)$

$= (\lg n)$

Analysis of Matrix Multiplication

Work: $M_1(n) = 8\,M_1(n/2) + \quad (n^2)$

$= \quad (n^3)$

Critical path: $M_\infty(n) = M_\infty(n/2) + \quad (\lg n)$

$= \quad (\lg^2 n)$

Parallelism: $\dfrac{M_1(n)}{M_\infty(n)} = \quad (n^3/\lg^2 n)$

For 1000 £ 1000 matrices, parallelism ¼ 10^7.

Stack Temporaries

```
cilk Mult(*C,*A,*B,n)
{ float T[n][n];
   h base case & partition matrices i
   spawn Mult(C11,A11,B11,n/2);
   spawn Mult(C12,A11,B12,n/2);
   spawn Mult(C22,A21,B12,n/2);
   spawn Mult(C21,A21,B11,n/2);
   spawn Mult(T11,A12,B21,n/2);
   spawn Mult(T12,A12,B22,n/2);
   spawn Mult(T22,A22,B22,n/2);
   spawn Mult(T21,A22,B21,n/2);
   sync;
   spawn Add(C,T,n);
   sync;
   return;
}
```

In modern hierarchical-memory microprocessors, memory accesses are so expensive that minimizing storage often yields higher performance.

No-Temp Matrix Multiplication

```
cilk Mult2(*C,*A,*B,n
{ // C = C + A * B
  h base case & partition matrices i
  spawn Mult2(C11,A11,B11,n/2);
  spawn Mult2(C12,A11,B12,n/2);
  spawn Mult2(C22,A21,B12,n/2);
  spawn Mult2(C21,A21,B11,n/2);
  sync;
  spawn Mult2(C21,A22,B21,n/2);
  spawn Mult2(C22,A22,B22,n/2);
  spawn Mult2(C12,A12,B22,n/2);
  spawn Mult2(C11,A12,B21,n/2);
  sync;
  return;
}
```

Saves space at the expense of critical path.

Analysis of No-Temp Multiply

Work: $M_1(n) = \quad (n^3)$

Critical path: $M_\infty(n) = 2\,M_\infty(n/2) + \quad (1)$

$= \quad (n)$

Parallelism: $\dfrac{M_1(n)}{M_\infty(n)} = \quad (n^2)$

For 1000 £ 1000 matrices, parallelism ¼ 10^6.
Faster in practice.

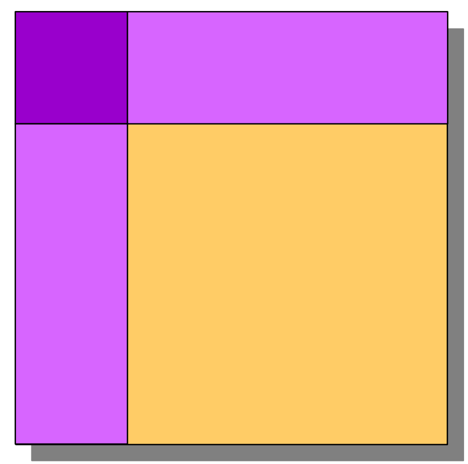
Ordinary Matrix Multiplication

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

IDEA: Spawn n^2 inner products in parallel. Compute each inner product in parallel.

Work: (n^3)
Critical path: $(\lg n)$
Parallelism: $(n^3/\lg n)$

BUT, this algorithm exhibits poor locality and does not exploit the cache hierarchy of modern microprocessors.



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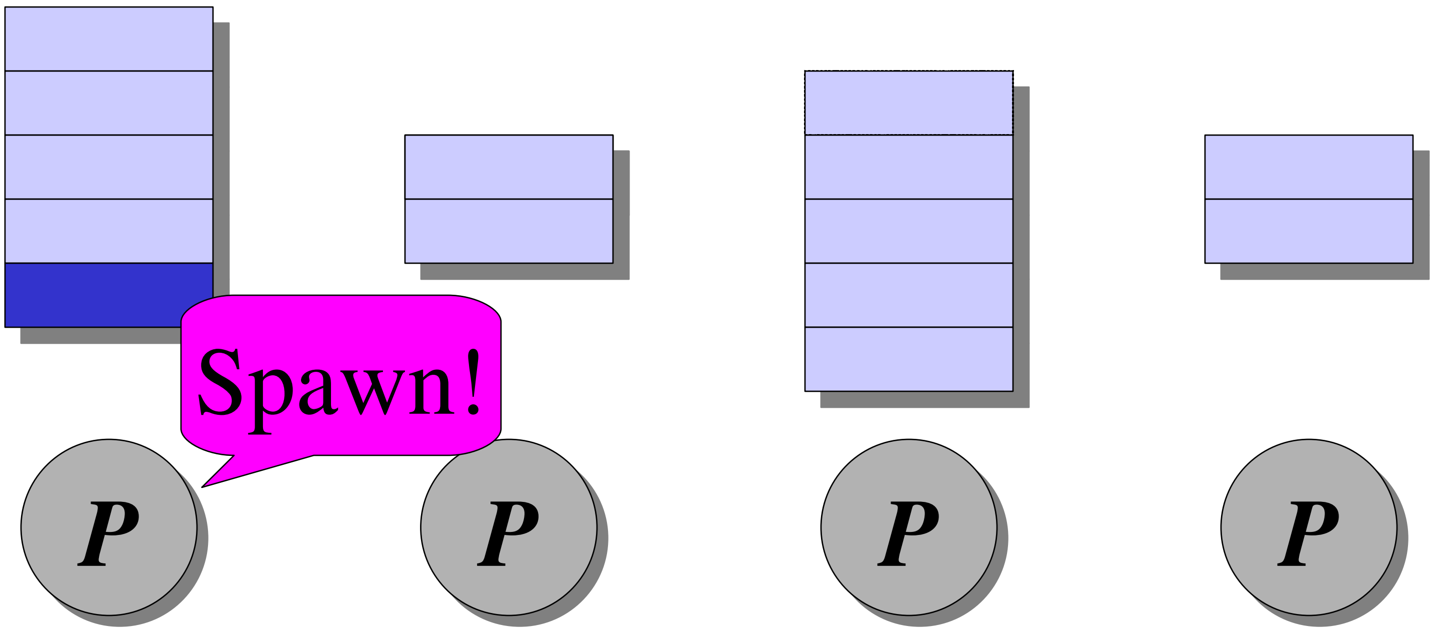
Cilk's Work-Stealing Scheduler

Each processor maintains a ***work deque*** of ready threads, and it manipulates the bottom of the deque like a stack.



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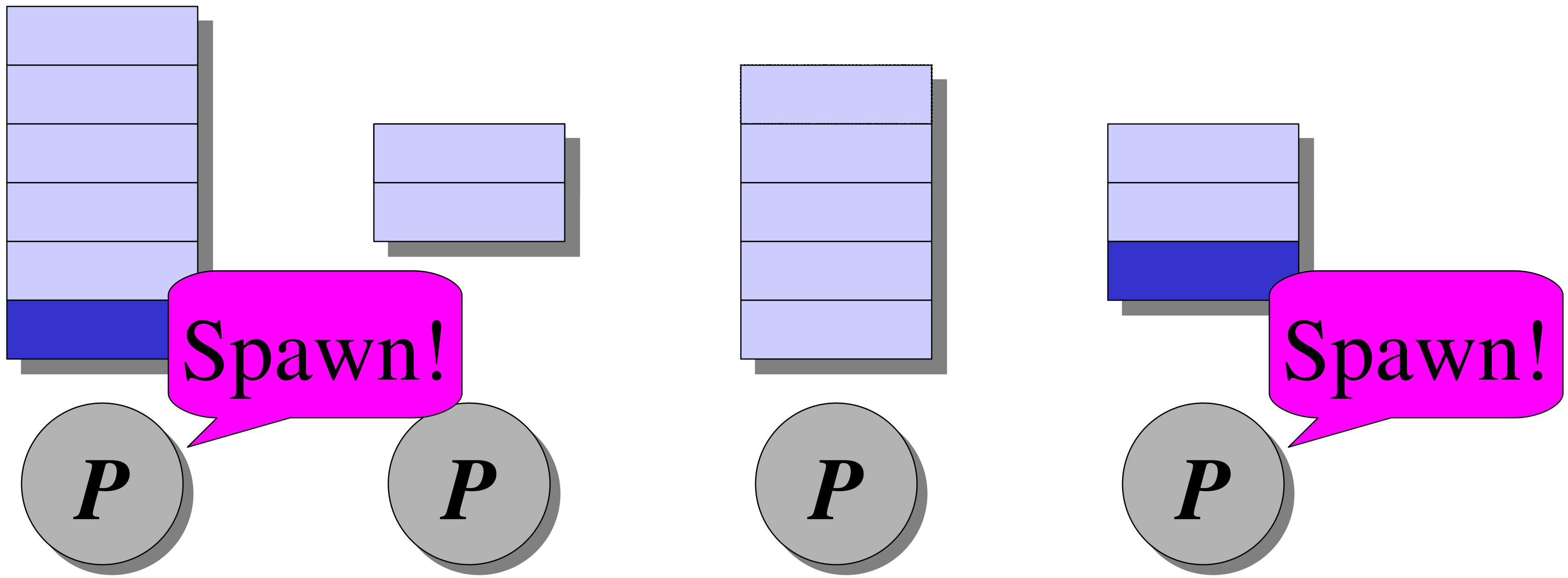

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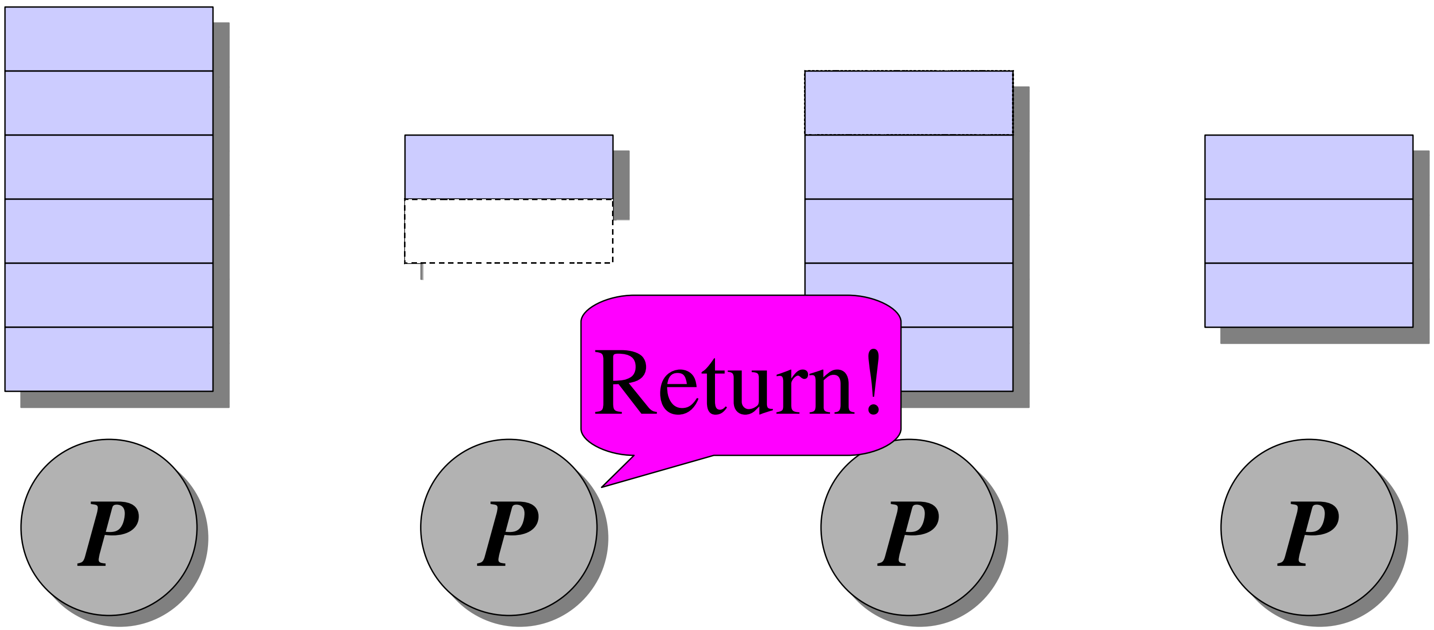

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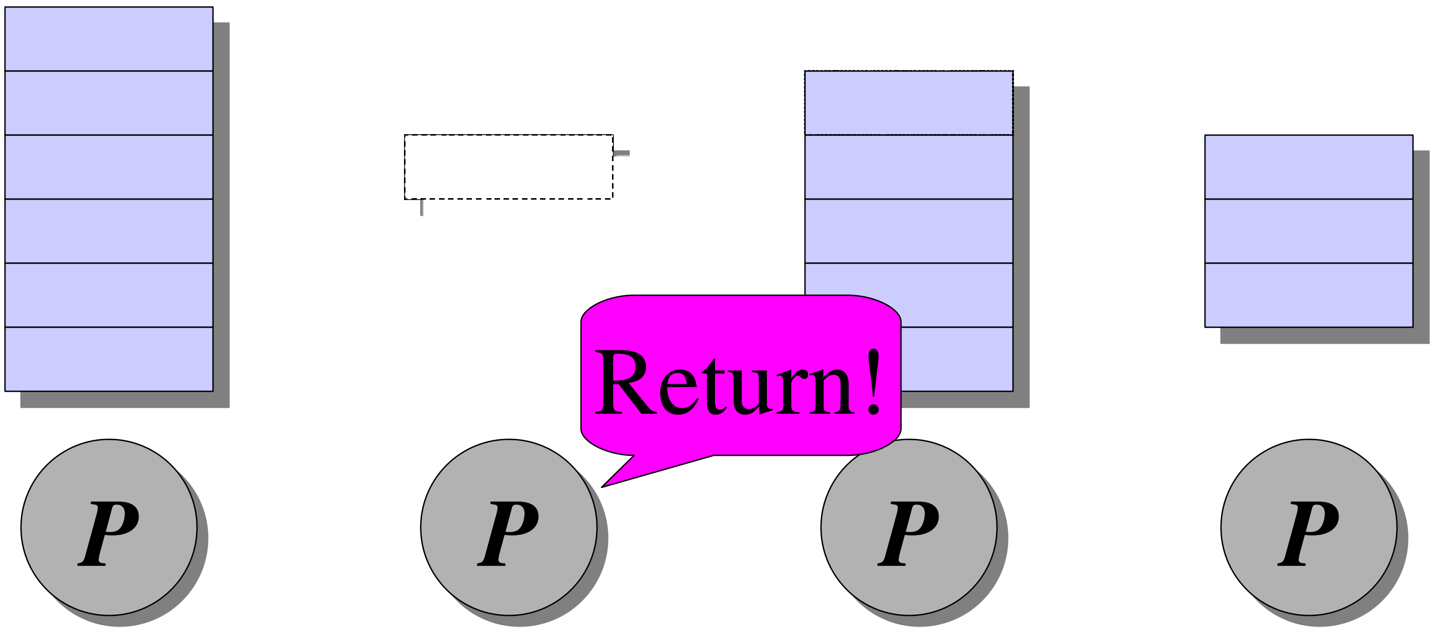

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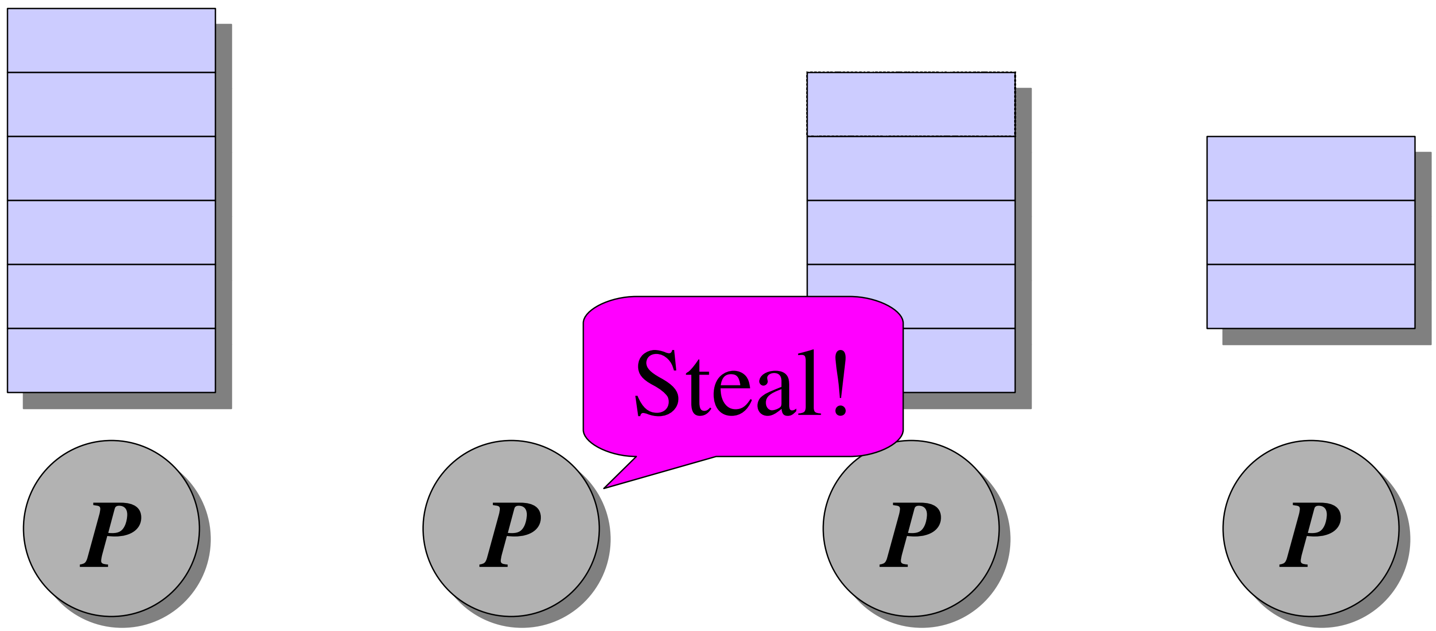
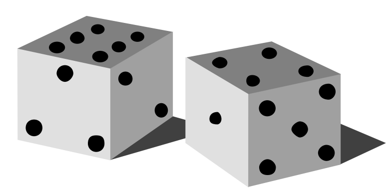


When a processor runs out of work, it ***steals*** a thread from the top of a ***random*** victim's deque.



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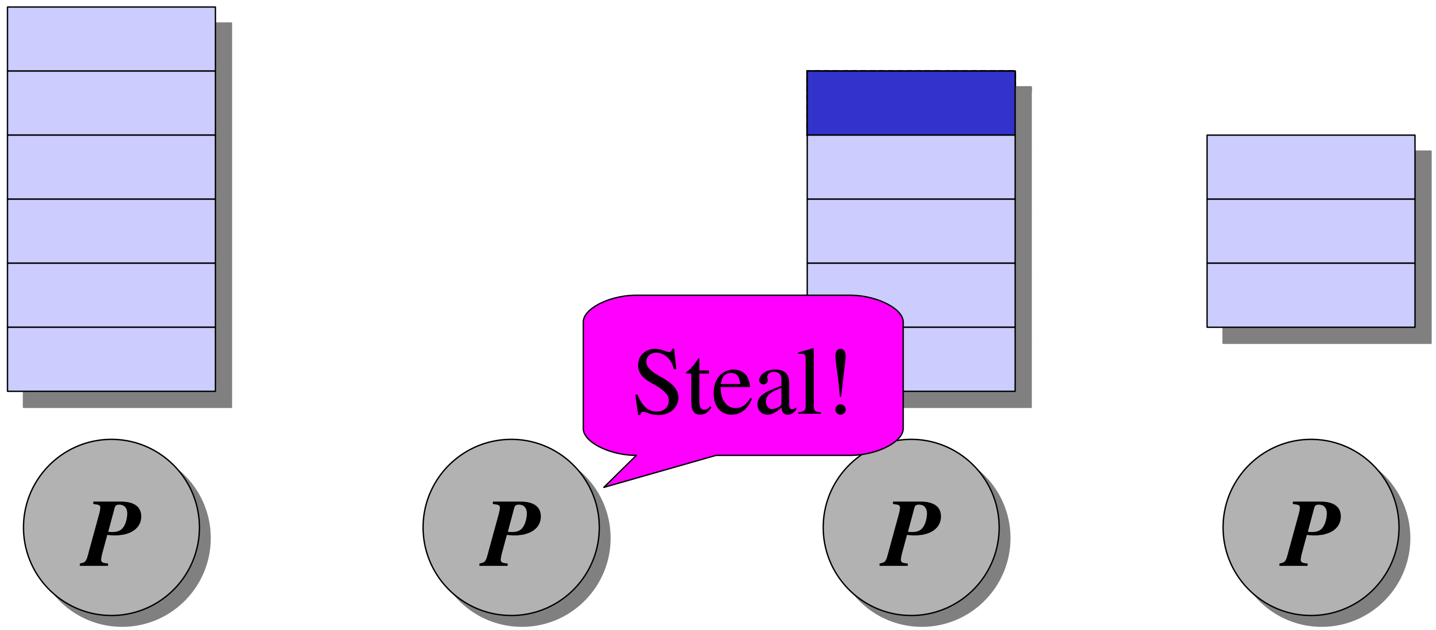

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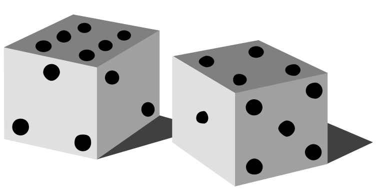

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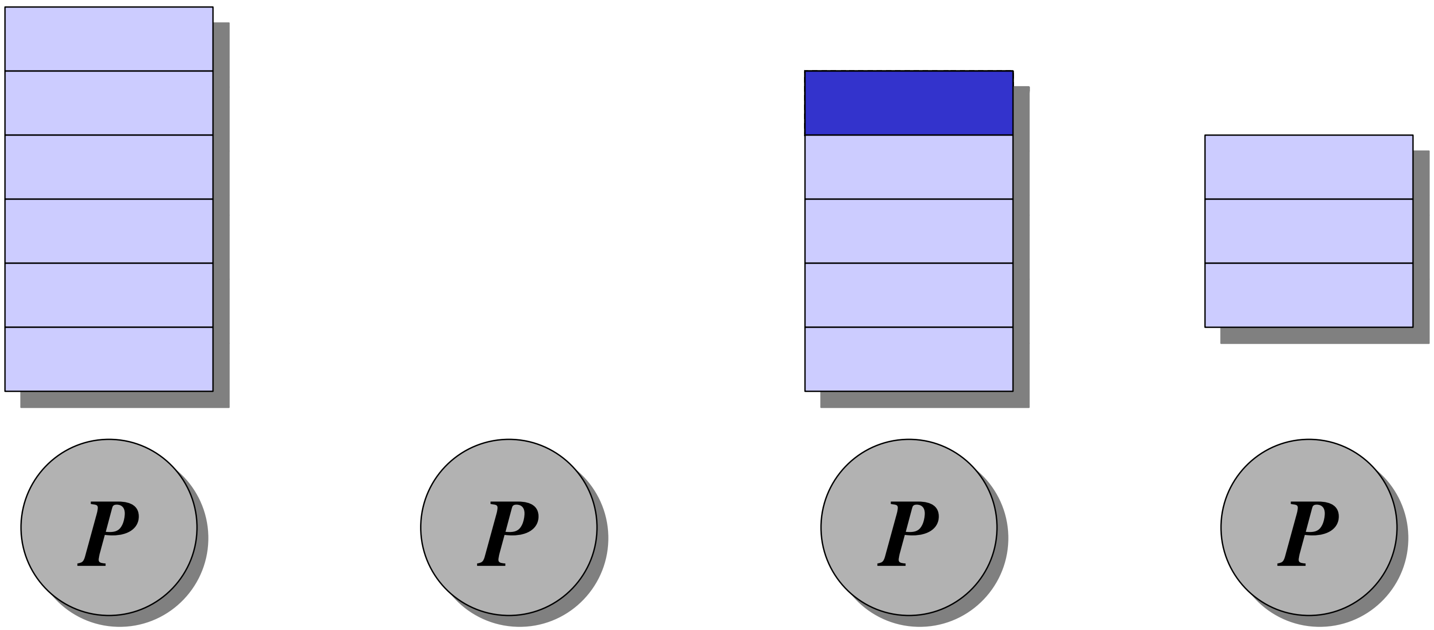

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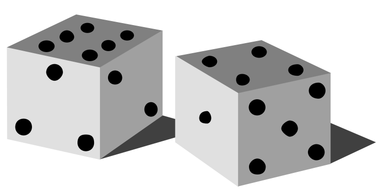

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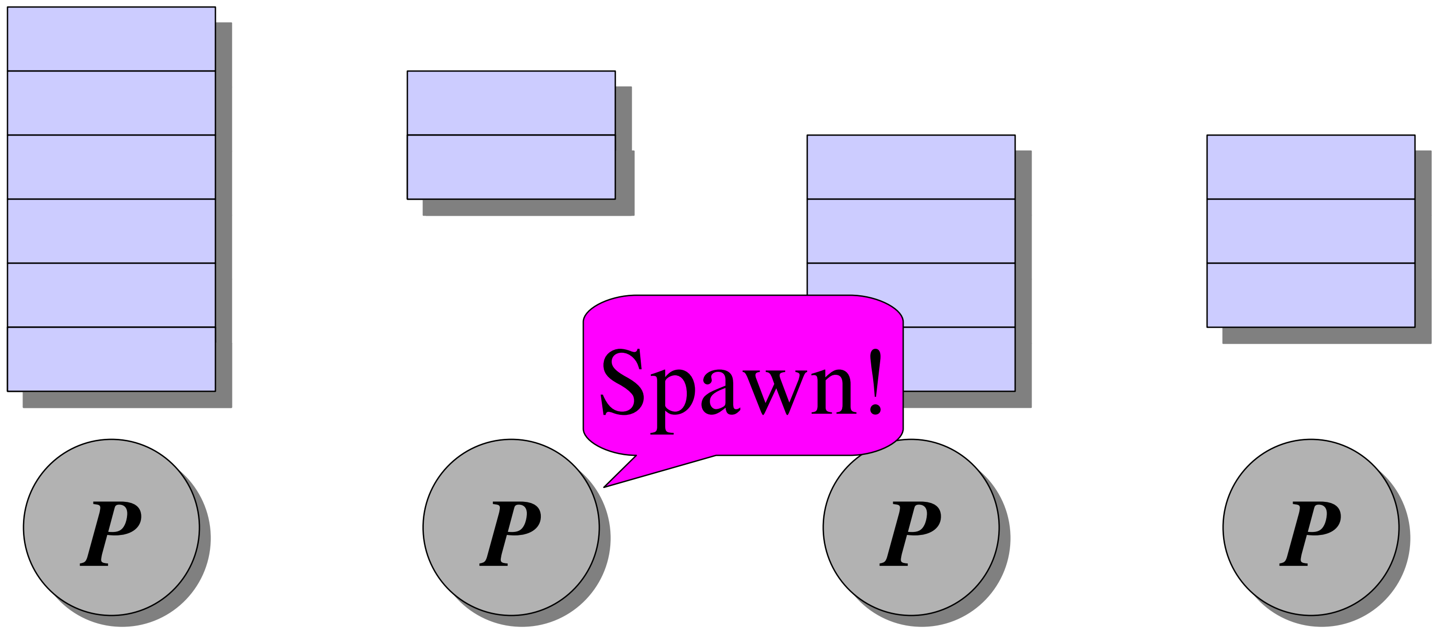

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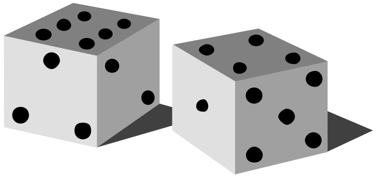

Performance of Work-Stealing

Theorem: A work-stealing scheduler achieves an expected running time of

$$T_P \le T_1/P + O(T_1)$$

on P processors.

Pseudoproof. A processor is either ***working*** or ***stealing***. The total time all processors spend working is T_1. Each steal has a $1/P$ chance of reducing the critical-path length by 1. Thus, the expected number of steals is $O(PT_1)$. Since there are P processors, the expected time is $(T_1 + O(PT_1))/P = T_1/P + O(T_1)$.

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Data Parallelism

- ☺ High level
- ☺ Intuitive
- ☺ Scales up

- ☹ Conversion costs
- ☹ Doesn't scale down
- ☹ Antithetical to caches
- ☹ Two-source problem
- ☹ Performance from tuned libraries

Example:

```
C = A + B;
D = A – B;
```

6 memory references, rather than 4.

Message Passing

- Scales up
- No compiler support needed
- Large inertia
- Runs anywhere

- Coarse grained
- Protocol intensive
- Difficult to debug
- Two-source problem
- Performance from tuned libraries

Shared memory ← *easier* In-core

harder ↓ ↓ *harder*

Distributed memory ← *easier* Out-of-core

Conventional (Persistent) Multithreading

- Scales up and down
- No compiler support needed
- Large inertia
- Evolutionary

- Clumsy
- No load balancing
- Coarse-grained control
- Protocol intensive
- Difficult to debug

Parallelism for *programs*, not *procedures*.

Dynamic Multithreading

- High-level linguistic support for fine-grained control and data manipulation.
- Algorithmic programming model based on work and critical path.
- Easy conversion from existing codes.
- Applications that scale up and down.
- Processor-oblivious machine model that can be implemented in an adaptively parallel fashion.
- Doesn't support a "program model" of parallelism.

Current Research

- We are currently designing ***jCilk***, a Java-based language that fuses dynamic and persistent multithreading in a single linguistic framework.
- A key piece of algorithmic technology is an ***adaptive task scheduler*** that guarantees fair and efficient execution.
- ***Hardware transactional memory*** appears to simplify thread synchronization and improve performance compared with locking.
- The *Nondeterminator 3* will be the first parallel data-race detector to guarantee both efficiency and linear speed-up.

Cilk Contributors

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…plus many MIT students and SourceForgers.

Image removed due to copyright restrictions. Silkworms.

World Wide Web

Cilk source code, programming examples, documentation, technical papers, tutorials, and up-to-date information can be found at:

http://supertech.csail.mit.edu/cilk

Download CILK Today!

Research Collaboration

Cilk is now being used at many universities for teaching and research:

MIT, Carnegie-Mellon, Yale, Texas, Dartmouth
Alabama, New Mexico, Tel Aviv, Singapore.

We need help in maintaining, porting, and enhancing Cilk's infrastructure, libraries, and application code base. If you are interested, send email to:

cilk-support@supertech.lcs.mit.edu

Warning: *We are not organized!*

Cilk-5 Benchmarks

Program	Size	T_1	T_∞	T_1/T_∞	T_1/T_S	T_8	T_1/T_8
blockedmul	1024	29.9	.0046	6783	1.05	4.29	7.0
notempmul	1024	29.7	.0156	1904	1.05	3.9	7.6
strassen	1024	20.2	.5662	36	1.01	3.54	5.7
queens	22	150.0	.0015	96898	0.99	18.8	8.0
cilksort*	4.1M	5.4	.0048	1125	1.21	0.9	6.0
knapsack	30	75.8	.0014	54143	1.03	9.5	8.0
lu	2048	155.8	.4161	374	1.02	20.3	7.7
cholesky*	1.02M	1427.0	3.4	420	1.25	208	6.9
heat	2M	62.3	.16	384	1.08	9.4	6.6
fft	1M	4.3	.002	2145	0.93	0.77	5.6
barnes-hut	65536	124.0	.15	853	1.02	16.5	7.5

All benchmarks were run on a Sun Enterprise 5000 SMP with 8 167-megahertz UltraSPARC processors. All times are in seconds, repeatable to within 10%.

Ease of Programming

	Original C	Cilk	SPLASH-2
lines	1861	2019	2959
lines	0	158	1098
`diff` *lines*	0	463	3741
T_1/T_8	1	7.5	7.2
T_1/T_S	1	1.024	1.099
T_S/T_8	1	7.3	6.6

Barnes-Hut application for 64K particles running on a 167-MHz Sun Enterprise 5000.

ICFP Programming Contest

- An 8-person Cilk team won **FIRST PRIZE** in the 1998 Programming Contest sponsored by the International Conference on Functional Programming.
- Our Cilk "***Pousse***" program was undefeated among the 49 entries. (Half the entries were coded in C.)
- Parallelizing our program to run on 4 processors took less than 1% of our effort, but it gave us more than a 3.5× performance advantage over our competitors.
- The ICFP Tournament Directors cited Cilk as ***"the superior programming tool of choice for discriminating hackers."***
- For details, see:

`http://supertech.lcs.mit.edu/~pousse`

Image removed due to copyright restrictions.
First prize ribbon drawing.

Whither Functional Programming?

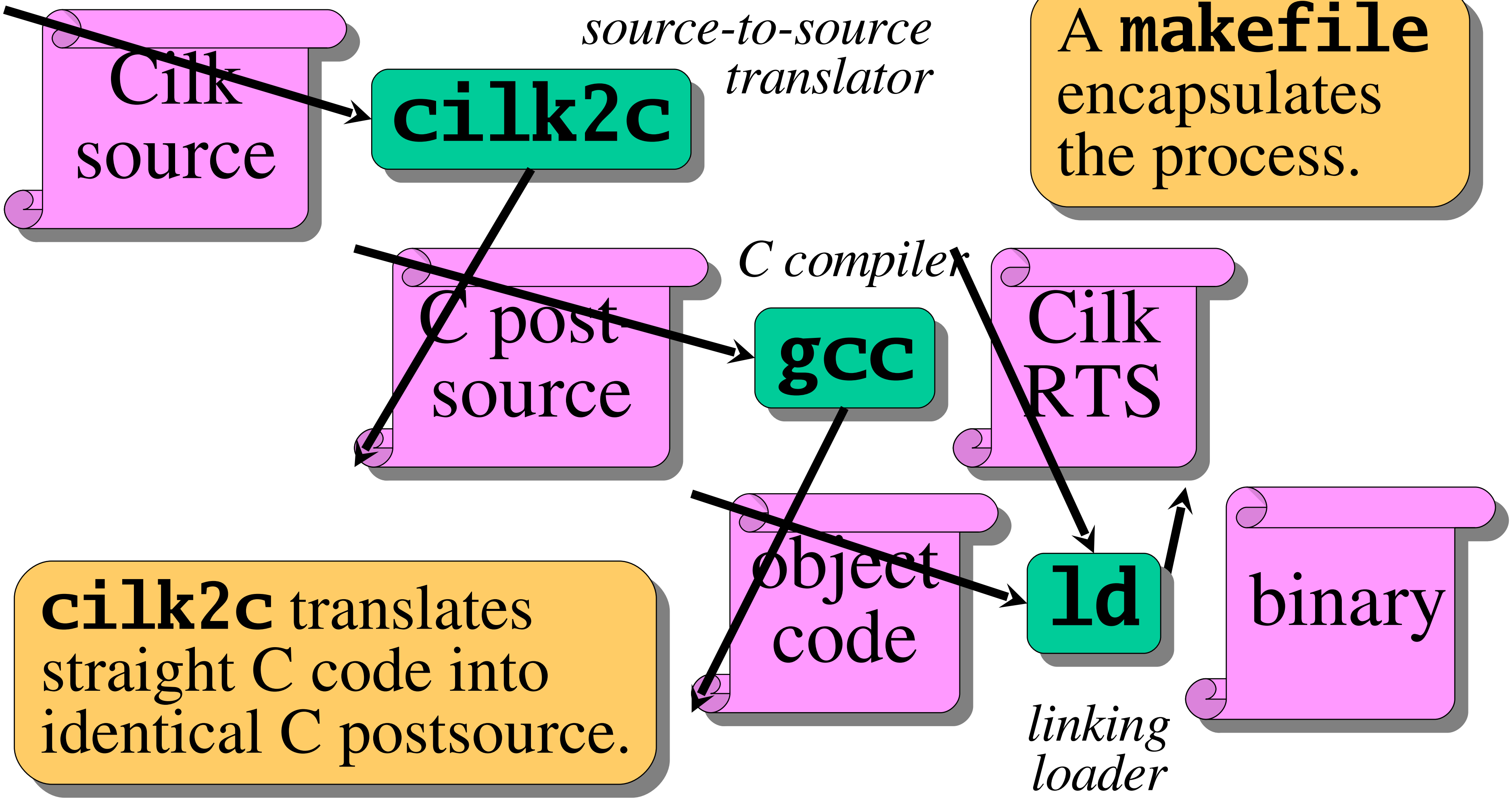
We have have had success using functional languages to generate high-performance portable C codes.

- **FFTW**: *The Fastest Fourier Transform in the West* [Frigo-Johnson 1997]: 2–5£ vendor libraries.
- Divide-and-conquer strategy optimizes cache use.
- A special-purpose compiler written in Objective CAML optimizes FFT dag for each recursive level.
- At runtime, FFTW measures the performance of various execution strategies and then uses dynamic programming to determine a good execution plan.

http://theory.lcs.mit.edu/~fftw

Sacred Cow

Compiling Cilk



A **makefile** encapsulates the process.

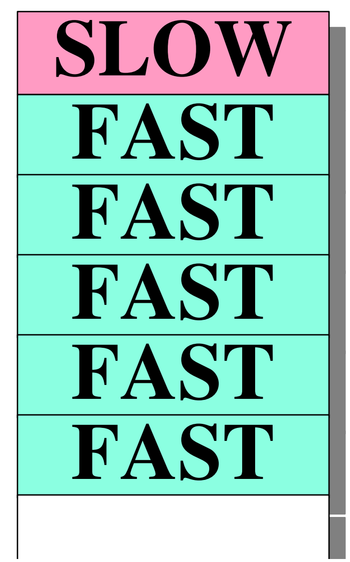
cilk2c translates straight C code into identical C postsource.

Cilk's Compiler Strategy

The **`cilk2c`** compiler generates two "clones" of each procedure:

- ***fast clone***—serial, common-case code.
- ***slow clone***—code with parallel bookkeeping.

- The *fast clone* is always spawned, saving live variables on Cilk's work deque (shadow stack).
- The *slow clone* is resumed if a thread is stolen, restoring variables from the shadow stack.
- A check is made whenever a procedure returns to see if the resuming parent has been stolen.



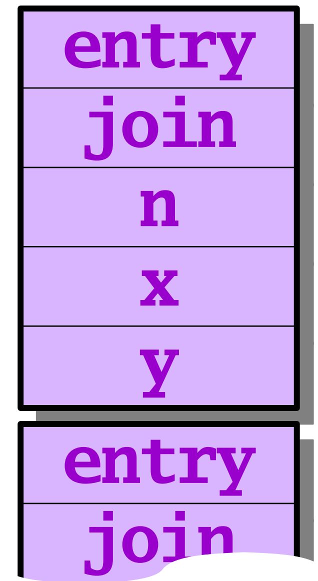
Compiling spawn (Fast Clone)

Cilk source

```
x = spawn fib(n-1);
```

cilk2c

frame



C post-source

```
frame->entry = 1;
frame->n = n;
push(frame);
```
} suspend parent

```
x = fib(n-1);
```
} run child

```
if (pop() == FAILURE)
  {   frame->x = x;
      frame->join--;
      h clean up & return
        to scheduler i      }
```
} resume parent remotely

Cilk deque

Compiling **sync** (Fast Clone)

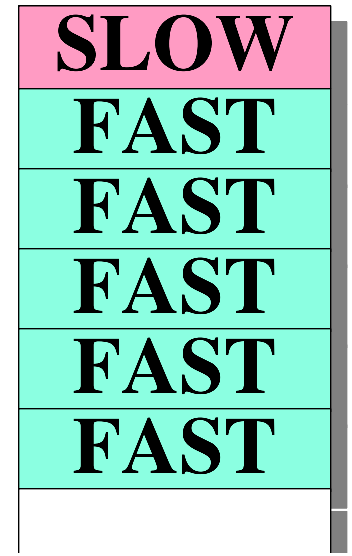
Cilk source

```
sync;
```

cilk2c

C post-source

```
;
```



No synchronization overhead in the fast clone!

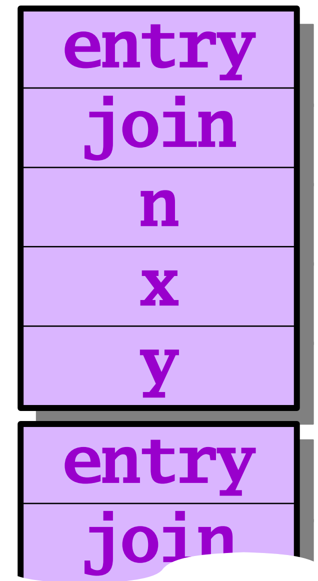
Compiling the Slow Clone

```
void fib_slow(fib_frame *frame)
{
  int n,x,y;
  switch (frame->entry) {
    case 1: goto L1;
    case 2: goto L2;
    case 3: goto L3;
    }
        . . .
  frame->entry = 1;
  frame->n = n;
  push(frame);
  x = fib(n-1);
  if (pop() == FAILURE)
    {  frame->x = x;
       frame->join--;
       ⟨clean up & return
         to scheduler⟩     }

  if (0) {
     L1:;
     n = frame->n;
  }
        . . .
}
```

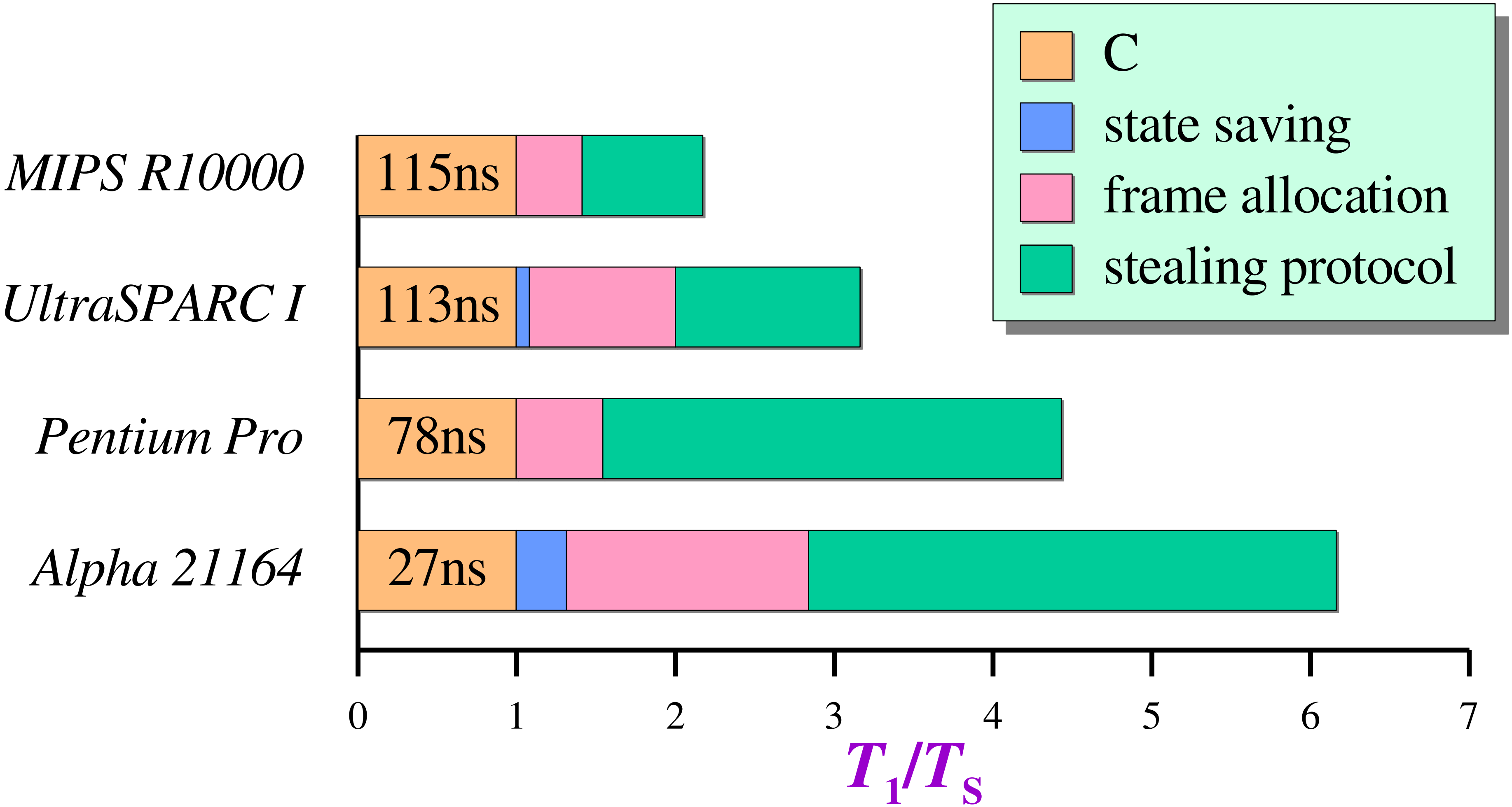
- restore program counter
- same as fast clone
- restore local variables if resuming
- continue

frame



Cilk deque

Breakdown of Work Overhead



***Benchmark*: `fib` on one processor.**

Mergesorting

```
cilk void Mergesort(int A[], int p, int r)
{
    int q;
    if ( p < r
        {
            q = (p+r)/2;
            spawn Mergesort(A,p,q);
            spawn Mergesort(A,q+1,r);
            sync;
            Merge(A,p,q,r);    // linear time
        }
}
```

$T_1(n) = 2\,T_1(n/2) + (n)$

$(n \lg n)$

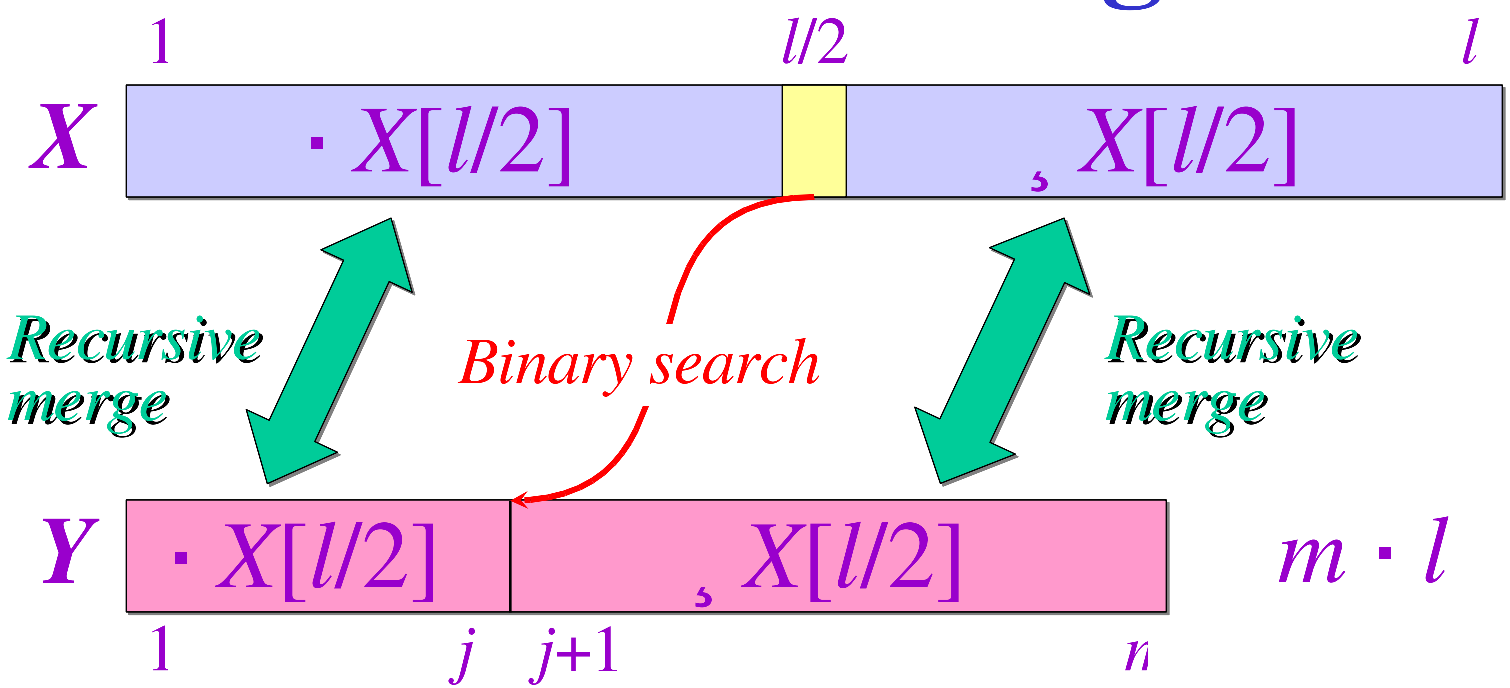
$T_\infty(n) = T_\infty(n/2) + (n)$

$= (n)$

Parallelism: PUNY!

$$\frac{(n \lg n)}{(n)} = (\lg n)$$

Parallel Merge



$T_1(n) = T_1(\quad n) + T_1((1- \quad)n) + \quad(\lg n)$, where $1/4 \cdot \qquad \cdot 3/4$

$= \quad (n)$

$T_\infty(n) = T_\infty(3n/4) + \quad (\lg n)$

$= \quad (\lg^2 n)$

Parallel Mergesort

$$T_1(n) = 2\,T_1(n/2) + \quad (n)$$
$$= \quad (n \lg n)$$

$$T_\infty(n) = T_\infty(n/2) + \quad (\lg^2 n)$$
$$= \quad (\lg^3 n)$$

Parallelism:

$$\frac{(n \lg n)}{(\lg^3 n)} = \quad (n/\lg^2 n)$$

- Our implementation of this algorithm yields a 21% work overhead and achieves a 6 times speedup on 8 processors (saturating the bus).
- Parallelism of $(n/\lg n)$ can be obtained at the cost of increasing the work by a constant factor.

Student Assignment

Implement the fastest 1000 £ 1000 matrix-multiplication algorithm.

- ***Winner*:** A variant of ***Strassen's algorithm*** which permuted the row-major input matrix into a bit-interleaved order before the calculation.
- ***Losers*:** Half the groups had ***race bugs***, because they didn't bother to run the Nondeterminator.
- ***Learners:*** Should have taught ***high-performance C*** programming first. The students spent most of their time optimizing the serial C code and little of their time Cilkifying it.

Caching Behavior

Cilk's scheduler guarantees that

$$Q_P/P \cdot Q_1/P + O(MT_\infty/B) ,$$

where Q_P is the total number of cache faults on P processors, each with a cache of size M and cache-line length B.

Divide-and-conquer "cache-oblivious" matrix multiplication has

$$Q_1(n) = O(1 + n^3/\sqrt{M}B) ,$$

which is asymptotically optimal.

IDEA: *Once a submatrix fits in cache, no further cache misses on its submatrices.*