Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.061 Introduction to Power Systems
Class Notes Chapter 10
Analytic Design Evaluation of Induction Machines *

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1 Introduction

Induction machines are perhaps the most widely used of all electric motors. They are generally simple to build and rugged, offer reasonable asynchronous performance: a manageable torque-speed curve, stable operation under load, and generally satisfactory efficiency. Because they are so widely used, they are worth understanding.

In addition to their current economic importance, induction motors and generators may find application in some new applications with designs that are not similar to motors currently in commerce. An example is very high speed motors for gas compressors, perhaps with squirrel cage rotors, perhaps with solid iron (or perhaps with both).

Because it is possible that future, high performance induction machines will be required to have characteristics different from those of existing machines, it is necessary to understand them from first principles, and that is the objective of this document. It starts with a circuit theoretical view of the induction machine. This analysis is strictly appropriate only for wound-rotor machines, but leads to an understanding of more complex machines. This model will be used to explain the basic operation of induction machines. Then we will derive a model for squirrel-cage machines. Finally, we will show how models for solid rotor and mixed solid rotor/squirrel cage machines can be constructed.

The view that we will take in this document is relentlessly classical. All of the elements that we will use are calculated from first principles, and we do not resort to numerical analysis or empirical methods unless we have no choice. While this may seem to be seriously limiting, it serves our basic objective here, which is to achieve an understanding of how these machines work. It is our feeling that once that understanding exists, it will be possible to employ more sophisticated methods of analysis to get more accurate results for those elements of the machines which do not lend themselves to simple analysis.

An elementary picture of the induction machine is shown in Figure 1. The rotor and stator are coaxial. The stator has a polyphase winding in slots. The rotor has either a winding or a cage, also

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in slots. This picture will be modified slightly when we get to talking of "solid rotor" machines, anon. Generally, this analysis is carried out assuming three phases. As with many systems, this generalizes to different numbers of phases with little difficulty.
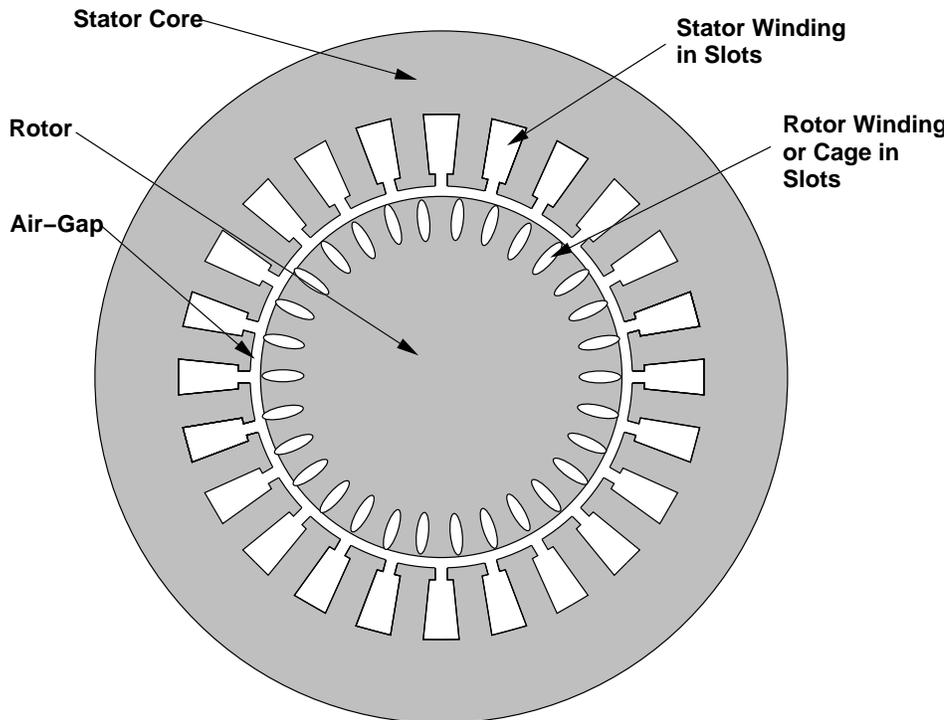

Figure 1: Axial View of an Induction Machine

2   Induction Motor Transformer Model

The induction machine has two electrically active elements: a rotor and a stator. In normal operation, the stator is excited by alternating voltage. (We consider here only polyphase machines). The stator excitation creates a magnetic field in the form of a rotating, or traveling wave, which induces currents in the circuits of the rotor. Those currents, in turn, interact with the traveling wave to produce torque. To start the analysis of this machine, assume that both the rotor and the stator can be described by balanced, three – phase windings. The two sets are, of course, coupled by mutual inductances which are dependent on rotor position. Stator fluxes are $(\lambda_a, \lambda_b, \lambda_c)$ and rotor fluxes are $(\lambda_A, \lambda_B, \lambda_C)$. The flux vs. current relationship is given by:

$$
\begin{bmatrix}
\lambda_a \\
\lambda_b \\
\lambda_c \\
\lambda_A \\
\lambda_B \\
\lambda_C
\end{bmatrix}
=
\begin{bmatrix}
\underline{L}_S & \underline{M}_{SR} \\
& \\
\underline{M}_{SR}^T & \underline{L}_R
\end{bmatrix}
\begin{bmatrix}
i_a \\
i_b \\
i_c \\
i_A \\
i_B \\
i_C
\end{bmatrix}
\tag{1}
$$

where the component matrices are:

$$\underline{\underline{L}}_S = \begin{bmatrix} L_a & L_{ab} & L_{ab} \\ L_{ab} & L_a & L_{ab} \\ L_{ab} & L_{ab} & L_a \end{bmatrix} \tag{2}$$

$$\underline{\underline{L}}_R = \begin{bmatrix} L_A & L_{AB} & L_{AB} \\ L_{AB} & L_A & L_{AB} \\ L_{AB} & L_{AB} & L_A \end{bmatrix} \tag{3}$$

The mutual inductance part of (1) is a circulant matrix:

$$\underline{\underline{M}}_{SR} = \begin{bmatrix} M\cos(p\theta) & M\cos(p\theta + \frac{2\pi}{3}) & M\cos(p\theta - \frac{2\pi}{3}) \\ M\cos(p\theta - \frac{2\pi}{3}) & M\cos(p\theta) & M\cos(p\theta + \frac{2\pi}{3}) \\ M\cos(p\theta + \frac{2\pi}{3}) & M\cos(p\theta - \frac{2\pi}{3}) & M\cos(p\theta) \end{bmatrix} \tag{4}$$

To carry the analysis further, it is necessary to make some assumptions regarding operation. To start, assume balanced currents in both the stator and rotor:

$$\begin{aligned} i_a &= I_S\cos(\omega t) \\ i_b &= I_S\cos(\omega t - \frac{2\pi}{3}) \\ i_c &= I_S\cos(\omega t + \frac{2\pi}{3}) \end{aligned} \tag{5}$$

$$\begin{aligned} i_A &= I_R\cos(\omega_R t + \xi_R) \\ i_B &= I_R\cos(\omega_R t + \xi_R - \frac{2\pi}{3}) \\ i_C &= I_R\cos(\omega_R t + \xi_R + \frac{2\pi}{3}) \end{aligned} \tag{6}$$

The rotor position θ can be described by

$$\theta = \omega_m t + \theta_0 \tag{7}$$

Under these assumptions, we may calculate the form of stator fluxes. As it turns out, we need only write out the expressions for λ_a and λ_A to see what is going on:

$$\begin{aligned} \lambda_a &= (L_a - L_{ab})I_s\cos(\omega t) + MI_R(\cos(\omega_R t + \xi_R)\cos p(\omega_m + \theta_0) \\ &+ \cos(\omega_R t + \xi_R + \frac{2\pi}{3})\cos(p(\omega_m t + \theta_0) - \frac{2\pi}{3}) + \cos(\omega_R t + \xi_R - \frac{2\pi}{3})\cos(p(\omega_m t + \theta_0) + \frac{2\pi}{3}) \end{aligned} \tag{8}$$

which, after reducing some of the trig expressions, becomes:

$$\lambda_a = (L_a - L_{ab})I_s\cos(\omega t) + \frac{3}{2}MI_R\cos((p\omega_m + \omega_R)t + \xi_R + p\theta_0) \tag{9}$$

Doing the same thing for the rotor phase A yields:

$$\begin{aligned} \lambda_A &= MI_s(\cos p(\omega_m t + \theta_0)\cos(\omega t)) + \cos(p(\omega_m t + \theta_0) - \frac{2\pi}{3})\cos(\omega t - \frac{2\pi}{3}) \\ &+ \cos(p(\omega_m t + \theta_0) + \frac{2\pi}{3})\cos(\omega t + \frac{2\pi}{3}) + (L_A - L_{AB})I_R\cos(\omega_R t + \xi_R) \end{aligned} \tag{10}$$

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This last expression is, after manipulating:

$$\lambda_A = \frac{3}{2}MI_s \cos((\omega - p\omega_m)t - p\theta_0) + (L_A - L_{AB})I_R \cos(\omega_R t + \xi_R) \qquad (11)$$

These two expressions, 9 and 11 give expressions for fluxes in the armature and rotor windings in terms of currents in the same two windings, assuming that both current distributions are sinusoidal in time and space and represent balanced distributions. The next step is to make another assumption, that the stator and rotor frequencies match through rotor rotation. That is:

$$\omega - p\omega_m = \omega_R \qquad (12)$$

It is important to keep straight the different frequencies here:

ω is stator electrical frequency
ω_R is rotor electrical frequency
ω_m is mechanical rotation speed

so that $p\omega_m$ is *electrical* rotation speed.

To refer rotor quantities to the stator frame (i.e. non- rotating), and to work in complex amplitudes, the following definitions are made:

$$\lambda_a = Re(\underline{\Lambda}_a e^{j\omega t}) \qquad (13)$$

$$\lambda_A = Re(\underline{\Lambda}_A e^{j\omega_R t}) \qquad (14)$$

$$i_a = Re(\underline{I}_a e^{j\omega t}) \qquad (15)$$

$$i_A = Re(\underline{I}_A e^{j\omega_R t}) \qquad (16)$$

With these definitions, the complex amplitudes embodied in 58 and 66 become:

$$\underline{\Lambda}_a = L_S \underline{I}_a + \frac{3}{2}M\underline{I}_A e^{j(\xi_R + p\theta_0)} \qquad (17)$$

$$\underline{\Lambda}_A = \frac{3}{2}M\underline{I}_a e^{-jp\theta_0} + L_R \underline{I}_A e^{j\xi_R} \qquad (18)$$

There are two phase angles embedded in these expressions: θ_0 which describes the rotor physical phase angle with respect to stator current and ξ_R which describes phase angle of rotor currents with respect to stator currents. We hereby invent two new rotor variables:

$$\underline{\Lambda}_{AR} = \underline{\Lambda}_A e^{jp\theta)} \qquad (19)$$

$$\underline{I}_{AR} = \underline{I}_A e^{j(p\theta_0 + \xi_R)} \qquad (20)$$

These are rotor flux and current referred to armature phase angle. Note that $\underline{\Lambda}_{AR}$ and $\underline{I}_{AR}$ have the same phase relationship to each other as do $\underline{\Lambda}_A$ and $\underline{I}_A$. Using 19 and 20 in 17 and 18, the basic flux/current relationship for the induction machine becomes:

$$\begin{bmatrix} \underline{\Lambda}_a \\ \underline{\Lambda}_{AR} \end{bmatrix} = \begin{bmatrix} L_S & \frac{3}{2}M \\ \frac{3}{2}M & L_R \end{bmatrix} \begin{bmatrix} \underline{I}_a \\ \underline{I}_{AR} \end{bmatrix} \qquad (21)$$

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This is an equivalent single- phase statement, describing the flux/current relationship in phase a, assuming balanced operation. The same expression will describe phases b and c.

Voltage at the terminals of the stator and rotor (possibly equivalent) windings is, then:

$$\underline{V}_a = j\omega\underline{\Lambda}_a + R_a\underline{I}_a \tag{22}$$

$$\underline{V}_{AR} = j\omega_R\underline{\Lambda}_{AR} + R_A\underline{I}_{AR} \tag{23}$$

or:

$$\underline{V}_a = j\omega L_S\underline{I}_a + j\omega\frac{3}{2}M\underline{I}_{AR} + R_a\underline{I}_a \tag{24}$$

$$\underline{V}_{AR} = j\omega_R\frac{3}{2}M\underline{I}_a + j\omega_R L_R\underline{I}_{AR} + R_A\underline{I}_{AR} \tag{25}$$

To carry this further, it is necessary to go a little deeper into the machine's parameters. Note that L_S and L_R are synchronous inductances for the stator and rotor. These may be separated into space fundamental and "leakage" components as follows:

$$L_S = L_a - L_{ab} = \frac{3}{2}\frac{4}{\pi}\frac{\mu_0 R l N_S^2 k_S^2}{p^2 g} + L_{Sl} \tag{26}$$

$$L_R = L_A - L_{AB} = \frac{3}{2}\frac{4}{\pi}\frac{\mu_0 R l N_R^2 k_R^2}{p^2 g} + L_{Rl} \tag{27}$$

Where the normal set of machine parameters holds:

R is rotor radius
l is active length
g is the effective air- gap
p is the number of pole- pairs
N represents number of turns
k represents the winding factor
S as a subscript refers to the stator
R as a subscript refers to the rotor
L_l is "leakage" inductance

The two leakage terms L_{Sl} and L_{Rl} contain higher order harmonic stator and rotor inductances, slot inducances, end- winding inductances and, if necessary, a provision for rotor skew. Essentially, they are used to represent all flux in the rotor and stator that is not mutually coupled.

In the same terms, the stator- to- rotor mutual inductance, which is taken to comprise *only* a space fundamental term, is:

$$M = \frac{4}{\pi}\frac{\mu_0 R l N_S N_R k_S k_R}{p^2 g} \tag{28}$$

Note that there are, of course, space harmonic mutual flux linkages. If they were to be included, they would hair up the analysis substantially. We ignore them here and note that they do have an effect on machine behavior, but that effect is second- order.

Air- gap permeance is defined as:

$$\wp_{ag} = \frac{4}{\pi} \frac{\mu_0 R l}{p^2 g} \tag{29}$$

so that the inductances are:

$$L_S = \frac{3}{2} \wp_{ag} k_S^2 N_S^2 + L_{Sl} \tag{30}$$

$$L_R = \frac{3}{2} \wp_{ag} k_R^2 N_R^2 + L_{Rl} \tag{31}$$

$$M = \wp_{ag} N_S N_R k_S k_R \tag{32}$$

Here we define "slip" s by:

$$\omega_R = s\omega \tag{33}$$

so that

$$s = 1 - \frac{p\omega_m}{\omega} \tag{34}$$

Then the voltage balance equations become:

$$\underline{V}_a = j\omega \left(\frac{3}{2} \wp_{ag} k_S^2 N_S^2 + L_{Sl} \right) \underline{I}_a + j\omega \frac{3}{2} \wp_{ag} N_S N_R k_S k_R \underline{I}_{AR} + R_a \underline{I}_a \tag{35}$$

$$\underline{V}_{AR} = js\omega \frac{3}{2} \wp_{ag} N_S N_R k_S k_R \underline{I}_a + js\omega \left(\frac{3}{2} \wp_{ag} k_R^2 N_R^2 + L_{Rl} \right) \underline{I}_{AR} + R_A \underline{I}_{AR} \tag{36}$$

At this point, we are ready to define rotor current referred to the stator. This is done by assuming an effective turns ratio which, in turn, defines an equivalent stator current to produce the same fundamental MMF as a given rotor current:

$$\underline{I}_2 = \frac{N_R k_R}{N_S k_S} \underline{I}_{AR} \tag{37}$$

Now, if we assume that the rotor of the machine is shorted so that $\underline{V}_{AR} = 0$ and do some manipulation we obtain:

$$\underline{V}_a = j(X_M + X_1)\underline{I}_a + jX_M \underline{I}_2 + R_a \underline{I}_a \tag{38}$$

$$0 = jX_M \underline{I}_a + j(X_M + X_2)\underline{I}_2 + \frac{R_2}{s} \underline{I}_2 \tag{39}$$

where the following definitions have been made:

$$X_M = \frac{3}{2} \omega \wp_{ag} N_S^2 k_S^2 \tag{40}$$

$$X_1 = \omega L_{Sl} \tag{41}$$

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$$X_2 = \omega L_{Rl} \left(\frac{N_S k_S}{N_R k_R} \right)^2 \tag{42}$$

$$R_2 = R_A \left(\frac{N_S k_S}{N_R k_R} \right)^2 \tag{43}$$

These expressions describe a simple equivalent circuit for the induction motorshow in Figure 2. We will amplify on this equivalent circuit anon.
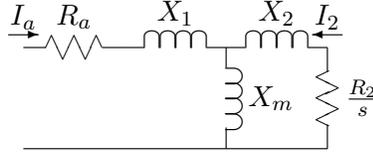

Figure 2: Equivalent Circuit

2.1 Effective Air-Gap: Carter's Coefficient

In induction motors, where the air-gap is usually quite small, it is necessary to correct the air-gap permeance for the effect of slot openings. These make the permeance of the air-gap slightly smaller than calculated from the physical gap, effectively making the gap a bit bigger. The ratio of effective to physical gap is:

$$g_{\text{eff}} = g \frac{t + s}{t + s - g f(\alpha)} \tag{44}$$

where

$$f(\alpha) = f \left(\frac{s}{2g} \right) = \alpha \tan(\alpha) - \log \sec \alpha \tag{45}$$

3 Operation: Energy Balance

Now we are ready to see how the induction machine actually works. Assume for the moment that Figure 2 represents one phase of a polyphase system and that the machine is operated under balanced conditions and that speed is constant or varying only slowly. "Balanced conditions" means that each phase has the same terminal voltage magnitude and that the phase difference between phases is a uniform. Under those conditions, we may analyze each phase separately (as if it were a single phase system). Assume an RMS voltage magnitude of V_t across each phase.

The "gap impedance", or the impedance looking to the right from the right-most terminal of X_1 is:

$$Z_g = j X_m || (j X_2 + \frac{R_2}{s}) \tag{46}$$

A total, or terminal impedance is then

$$Z_t = jX_1 + R_a + Z_g \tag{47}$$

and terminal current is

$$I_t = \frac{V_t}{Z_t} \tag{48}$$

Rotor current is found by using a current divider:

$$I_2 = I_t \frac{jX_m}{jX_2 + \frac{R_2}{s}} \tag{49}$$

"Air-gap" power is then calculated (assuming a three-phase machine):

$$P_{ag} = 3|I_2|^2 \frac{R_2}{s} \tag{50}$$

This is real (time-average) power crossing the air-gap of the machine. Positive slip implies rotor speed less than synchronous and positive air-gap power (motor operation). Negative slip means rotor speed is higher than synchronous, negative air-gap power (from the rotor to the stator) and generator operation.

Now, note that this equivalent circuit represents a real physical structure, so it should be possible to calculate power dissipated in the physical rotor resistance, and that is:

$$P_s = P_{ag}s \tag{51}$$

(Note that, since both P_{ag} and s will always have the same sign, dissipated power is positive.) The rest of this discussion is framed in terms of *motor* operation, but the conversion to *generator* operation is simple. The difference between power crossing the air-gap and power dissipated in the rotor resistance must be converted from mechanical form:

$$P_m = P_{ag} - P_s \tag{52}$$

and *electrical input* power is:

$$P_{in} = P_{ag} + P_a \tag{53}$$

where armature dissipation is:

$$P_a = 3|I_t|^2 R_a \tag{54}$$

Output (mechanical) power is

$$P_{\text{out}} = P_{ag} - P_w \tag{55}$$

Where P_w describes friction, windage and certain stray losses which we will discuss later.

And, finally, efficiency and power factor are:

$$\eta = \frac{P_{\text{out}}}{P_{in}} \tag{56}$$

$$\cos\psi = \frac{P_{\text{in}}}{3V_t I_t} \tag{57}$$

```
% --------------------------------------------------------
% Torque-Speed Curve for an Induction Motor
% Assumes the classical model
% This is a single-circuit model
% Required parameters are R1, X1, X2, R2, Xm, Vt, Ns
% Assumed is a three-phase motor
% This thing does a motoring, full speed range curve
% Copyright 1994 James L. Kirtley Jr.
% --------------------------------------------------------
s = .002:.002:1;                    % vector of slip
N = Ns .* (1 - s);                  % Speed, in RPM
oms = 2*pi*Ns/60;                   % Synchronous speed
Rr = R2 ./ s;                       % Rotor resistance
Zr = j*X2 + Rr;                     % Total rotor impedance
Za = par(j*Xm, Zr);                 % Air-gap impedance
Zt = R1 + j*X1 +Za;                 % Terminal impedance
Ia = Vt ./ Zt;                      % Terminal Current
I2 = Ia .* cdiv (Zr, j*Xm);         % Rotor Current
Pag = 3 .* abs(I2) .^2 .* Rr;       % Air-Gap Power
Pm = Pag .* (1 - s);                % Converted Power
Trq = Pag ./ oms;                   % Developed Torque
subplot(2,1,1)
plot(N, Trq)
title('Induction Motor');
ylabel('N-m');
subplot(2,1,2)
plot(N, Pm);
ylabel('Watts');
xlabel('RPM');
```

3.1   Example of Operation

The following MATLAB script generates a torque-speed and power-speed curve for the simple induction motor model described above. Note that, while the analysis does *not* require that any of the parameters, such as rotor resistance, be independent of rotor speed, this simple script does assume that all parameters are constant.

3.2   Example

That MATLAB script has been run for a standard motor with parameters given in Table 1.

Torque vs. speed and power vs. speed are plotted for this motor in Figure 3. These curves were generated by the MATLAB script shown above.

Table 1: Example, Standard Motor

Rating	300	kw
Voltage	440	VRMS, l-l
	254	VRMS, l-n
Stator Resistance R1	.0073	Ω
Rotor Resistance R2	.0064	Ω
Stator Reactance X1	.06	Ω
Rotor Reactance X2	.06	Ω
Magnetizing Reactance Xm	2.5	Ω
Synchronous Speed Ns	1200	RPM

4   Squirrel Cage Machine Model

Now we derive a circuit model for the squirrel-cage motor using field analytical techniques. The model consists of two major parts. The first of these is a description of stator flux in terms of stator and rotor currents. The second is a description of rotor current in terms of air- gap flux. The result of all of this is a set of expressions for the elements of the circuit model for the induction machine.

To start, assume that the rotor is symmetrical enough to carry a surface current, the fundamental of which is:

$$\overline{K}_r = \overline{\imath}_z Re\left(\underline{K}_r e^{j(s\omega t - p\phi')}\right)$$
$$= \overline{\imath}_z Re\left(\underline{K}_r e^{j(\omega t - p\phi)}\right) \tag{58}$$

Note that in 58 we have made use of the simple transformation between rotor and stator coordinates:

$$\phi' = \phi - \omega_m t \tag{59}$$

and that

$$p\omega_m = \omega - \omega_r = \omega(1 - s) \tag{60}$$

Here, we have used the following symbols:

$\underline{K}_r$ is complex amplitude of rotor surface current
s is per- unit "slip"
ω is stator electrical frequency
ω_r is rotor electrical frequency
ω_m is rotational speed

The rotor current will produce an air- gap flux density of the form:

$$B_r = Re\left(\underline{B}_r e^{j(\omega t - p\phi)}\right) \tag{61}$$

where

$$\underline{B}_r = -j\mu_0 \frac{R}{pg}\underline{K}_r \tag{62}$$
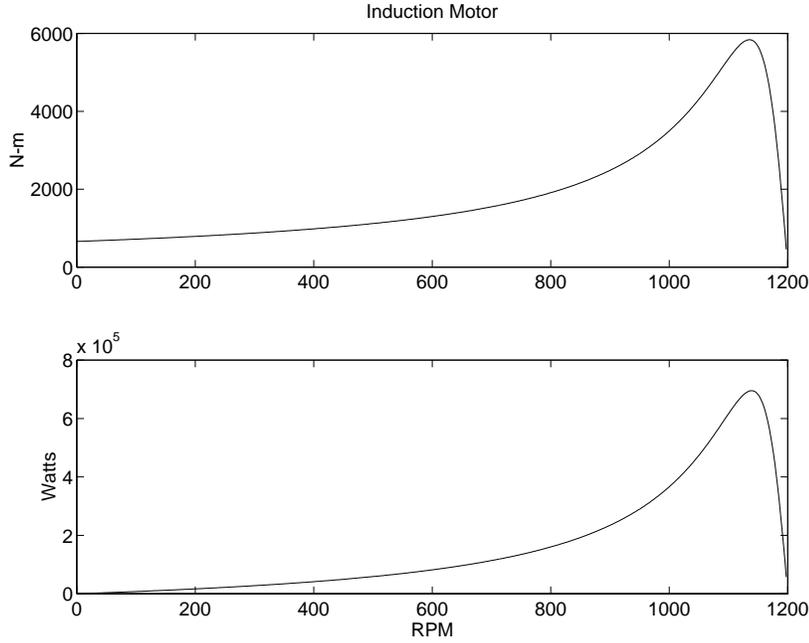
Figure 3: Torque and Power vs. Speed for Example Motor

Note that this describes only radial magnetic flux density produced by the space fundamental of *rotor* current. Flux linked by the armature winding due to this flux density is:

$$\lambda_{AR} = l N_S k_S \int_{-\frac{\pi}{p}}^{0} B_r(\phi) R d\phi \tag{63}$$

This yields a complex amplitude for λ_{AR}:

$$\lambda_{AR} = Re\left(\underline{\Lambda}_{AR} e^{j\omega t}\right) \tag{64}$$

where

$$\underline{\Lambda}_{AR} = \frac{2l\mu_0 R^2 N_S k_S}{p^2 g} \underline{K}_r \tag{65}$$

Adding this to flux produced by the stator currents, we have an expression for total stator flux:

$$\underline{\Lambda}_a = \left(\frac{3}{2}\frac{4}{\pi}\frac{\mu_0 N_S^2 R l k_S^2}{p^2 g} + L_{Sl}\right)\underline{I}_a + \frac{2l\mu_0 R^2 N_S k_S}{p^2 g}\underline{K}_r \tag{66}$$

Expression 66 motivates a definiton of an equivalent rotor current I_2 in terms of the space fundamental of rotor surface current density:

$$\underline{I}_2 = \frac{\pi}{3}\frac{R}{N_S k_S}\underline{K}_z \tag{67}$$

Then we have the simple expression for stator flux:

$$\underline{\Lambda}_a = (L_{ad} + L_{Sl})\underline{I}_a + L_{ad}\underline{I}_2 \tag{68}$$

where L_{ad} is the fundamental space harmonic component of stator inductance:

$$L_{ad} = \frac{3}{2}\frac{4}{\pi}\frac{\mu_0 N_S^2 k_S^2 Rl}{p^2 g} \tag{69}$$

The second part of this derivation is the equivalent of finding a relationship between rotor flux and I_2. However, since this machine has no discrete windings, we must focus on the individual rotor bars.

Assume that there are N_R slots in the rotor. Each of these slots is carrying some current. If the machine is symmetrical and operating with balanced currents, we may write an expression for current in the kth slot as:

$$i_k = Re\left(\underline{I}_k e^{js\omega t}\right) \tag{70}$$

where

$$\underline{I}_k = \underline{I} e^{-j\frac{2\pi p}{N_R}} \tag{71}$$

and $\underline{I}$ is the complex amplitude of current in slot number zero. Expression 71 shows a uniform progression of rotor current *phase* about the rotor. All rotor slots carry the same current, but that current is phase retarded (delayed) from slot to slot because of relative rotation of the current wave at slip frequency.

The rotor current density can then be expressed as a sum of impulses:

$$K_z = Re\left(\sum_{k=0}^{N_R-1}\frac{1}{R}\underline{I}e^{j(\omega_r t - k\frac{2\pi p}{N_R})}\delta(\phi' - \frac{2\pi k}{N_R})\right) \tag{72}$$

The unit impulse function $\delta()$ is our way of approximating the rotor current as a series of impulsive currents around the rotor.

This rotor surface current may be expressed as a fourier series of traveling waves:

$$K_z = Re\left(\sum_{n=-\infty}^{\infty}\underline{K}_n e^{j(\omega_r t - np\phi')}\right) \tag{73}$$

Note that in 73, we are allowing for negative values of the space harmonic index n to allow for reverse- rotating waves. This is really part of an expansion in both time and space, although we are considering only the time fundamental part. We may recover the nth space harmonic component of 73 by employing the following formula:

$$\underline{K}_n = <\frac{1}{\pi}\int_0^{2\pi} K_r(\phi, t)e^{-j(\omega_r t - np\phi)}d\phi> \tag{74}$$

Here the brackets $<>$ denote time average and are here beause of the two- dimensional nature of the expansion. To carry out 74 on 72, first expand 72 into its complex conjugate parts:

$$K_r = \frac{1}{2}\sum_{k=0}^{N_R-1}\left\{\frac{\underline{I}}{R}e^{j(\omega_r t - k\frac{2\pi p}{N_R})} + \frac{\underline{I}^*}{R}e^{-j(\omega_r t - k\frac{2\pi p}{N_R})}\right\}\delta(\phi' - \frac{2\pi k}{N_R}) \tag{75}$$

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If 75 is used in 74, the second half of 75 results in a sum of terms which time average to zero. The first half of the expression results in:

$$\underline{K}_n = \frac{I}{2\pi R} \int_0^{2\pi} \sum_{k=0}^{N_R-1} e^{-j\frac{2\pi pk}{N_R}} e^{jnp\phi} \delta(\phi - \frac{2\pi k}{N_R}) d\phi \tag{76}$$

The impulse function turns the integral into an evaluation of the rest of the integrand at the impulse. What remains is the sum:

$$\underline{K}_n = \frac{I}{2\pi R} \sum_{k=0}^{N_R-1} e^{j(n-1)\frac{2\pi kp}{N_R}} \tag{77}$$

The sum in 77 is easily evaluated. It is:

$$\sum_{k=0}^{N_R-1} e^{j\frac{2\pi kp(n-1)}{N_R}} = \begin{cases} N_R & \text{if } (n-1)\frac{P}{N_R} = \text{integer} \\ 0 & \text{otherwise} \end{cases} \tag{78}$$

The integer in 78 may be positive, negative or zero. As it turns out, only the first three of these (zero, plus and minus one) are important, because these produce the largest magnetic fields and therefore fluxes. These are:

$$
\begin{aligned}
(n-1)\frac{p}{N_R} &= -1 & \text{or } n &= -\frac{N_R-p}{p} \\
&= 0 & \text{or } n &= 1 \\
&= 1 & \text{or } n &= \frac{N_R+p}{p}
\end{aligned} \tag{79}
$$

Note that 79 appears to produce space harmonic orders that may be of non- integer order. This is not really true: is is necessary that np be an integer, and 79 will always satisfy that condition.

So, the harmonic orders of interest to us are one and

$$n_+ = \frac{N_R}{p} + 1 \tag{80}$$

$$n_- = -\left(\frac{N_R}{p} - 1\right) \tag{81}$$

Each of the space harmonics of the squirrel- cage current will produce radial flux density. A surface current of the form:

$$K_n = Re\left(\frac{N_R\underline{I}}{2\pi R} e^{j(\omega_r t - np\phi')}\right) \tag{82}$$

produces radial magnetic flux density:

$$B_{rn} = Re\left(\underline{B}_{rn} e^{j(\omega_r t - np\phi')}\right) \tag{83}$$

where

$$\underline{B}_{rn} = -j\frac{\mu_0 N_R \underline{I}}{2\pi npg} \tag{84}$$

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In turn, each of the components of radial flux density will produce a component of induced voltage. To calculate that, we must invoke Faraday's law:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{85}$$

The radial component of 85, assuming that the fields do not vary with z, is:

$$\frac{1}{R}\frac{\partial}{\partial \phi}E_z = -\frac{\partial B_r}{\partial t} \tag{86}$$

Or, assuming an electric field component of the form:

$$E_{zn} = Re\left(\underline{E}_n e^{j(\omega_r t - np\phi)}\right) \tag{87}$$

Using 84 and 87 in 86, we obtain an expression for electric field induced by components of air-gap flux:

$$\underline{E}_n = \frac{\omega_r R}{np}\underline{B}_n \tag{88}$$

$$\underline{E}_n = -j\frac{\mu_0 N_R \omega_r R}{2\pi g(np)^2}\underline{I} \tag{89}$$

Now, the total voltage induced in a slot pushes current through the conductors in that slot. We may express this by:

$$\underline{E}_1 + \underline{E}_{n-} + \underline{E}_{n+} = \underline{Z}_{slot}\underline{I} \tag{90}$$

Now: in 90, there are three components of air- gap field. E_1 is the space fundamental field, produced by the space fundamental of rotor current as well as by the space fundamental of stator current. The other two components on the left of 90 are produced only by rotor currents and actually represent additional reactive impedance to the rotor. This is often called *zigzag* leakage inductance. The parameter Z_{slot} represents impedance of the slot itself: resistance and reactance associated with cross- slot magnetic fields. Then 90 can be re-written as:

$$\underline{E}_1 = \underline{Z}_{slot}\underline{I} + j\frac{\mu_0 N_R \omega_r R}{2\pi g}\left(\frac{1}{(n_+p)^2} + \frac{1}{(n_-p)^2}\right)\underline{I} \tag{91}$$

To finish this model, it is necessary to translate 91 back to the stator. See that 67 and 77 make the link between $\underline{I}$ and $\underline{I}_2$:

$$\underline{I}_2 = \frac{N_R}{6N_S k_S}\underline{I} \tag{92}$$

Then the electric field at the surface of the rotor is:

$$\underline{E}_1 = \left[\frac{6N_S k_S}{N_R}\underline{Z}_{slot} + j\omega_r\frac{3}{\pi}\frac{\mu_0 N_S k_S R}{g}\left(\frac{1}{(n_+p)^2} + \frac{1}{(n_-p)^2}\right)\right]\underline{I}_2 \tag{93}$$

This must be translated into an equivalent stator voltage. To do so, we use 88 to translate 93 into a statement of radial magnetic field, then find the flux liked and hence stator voltage from that. Magnetic flux density is:

$$\begin{aligned} \underline{B}_r &= \frac{p\underline{E}_1}{\omega_r R} \\ &= \left[\frac{6N_S k_S p}{N_R R} \left(\frac{R_{slot}}{\omega_r} + jL_{slot} \right) + j\frac{3}{\pi} \frac{\mu_0 N_S k_S p}{g} \left(\frac{1}{(n_+ p)^2} + \frac{1}{(n_- p)^2} \right) \right] \underline{I}_2 \end{aligned} \tag{94}$$

where the slot impedance has been expressed by its real and imaginary parts:

$$\underline{Z}_{slot} = R_{slot} + j\omega_r L_{slot} \tag{95}$$

Flux linking the armature winding is:

$$\lambda_{ag} = N_S k_S l R \int_{-\frac{\pi}{2p}}^{0} Re\left(\underline{B}_r e^{j(\omega t - p\phi)} \right) d\phi \tag{96}$$

Which becomes:

$$\lambda_{ag} = Re\left(\underline{\Lambda}_{ag} e^{j\omega t} \right) \tag{97}$$

where:

$$\underline{\Lambda}_{ag} = j\frac{2N_S k_S l R}{p} \underline{B}_r \tag{98}$$

Then "air- gap" voltage is:

$$\begin{aligned} \underline{V}_{ag} &= j\omega \underline{\Lambda}_{ag} = -\frac{2\omega N_S k_S l R}{p} \underline{B}_r \\ &= -\underline{I}_2 \left[\frac{12l N_S^2 k_S^2}{N_R} \left(j\omega L_{slot} + \frac{R_2}{s} \right) + j\omega \frac{6}{\pi} \frac{\mu_0 R l N_S^2 k_S^2}{g} \left(\frac{1}{(n_+ p)^2} + \frac{1}{(n_- p)^2} \right) \right] \end{aligned} \tag{99}$$

Expression 99 describes the relationship between the space fundamental air- gap voltage $\underline{V}_{ag}$ and rotor current $\underline{I}_2$. This expression fits the equivalent circuit of Figure 4 if the definitions made below hold:
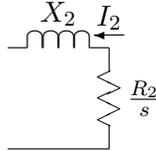

Figure 4: Rotor Equivalent Circuit

$$X_2 = \omega \frac{12l N_S^2 k_S^2}{N_R} L_{\text{slot}} + \omega \frac{6}{\pi} \frac{\mu_0 R l N_S^2 k_S^2}{g} \left(\frac{1}{(N_R + p)^2} + \frac{1}{(N_R - p)^2} \right) \tag{100}$$

$$R_2 = \frac{12l N_S^2 k_S^2}{N_R} R_{slot} \tag{101}$$

The first term in 100 expresses slot leakage inductance for the rotor. Similarly, 101 expresses rotor resistance in terms of slot resistance. Note that L_{slot} and R_{slot} are both expressed per unit length. The second term in 100 expresses the "zigzag" leakage inductance resulting from harmonics on the order of rotor slot pitch.

Next, see that armature flux is just equal to air- gap flux plus armature leakage inductance. That is, 68 could be written as:

$$\underline{\Lambda}_a = \underline{\Lambda}_{ag} + L_{al}\underline{I}_a \tag{102}$$

There are a number of components of stator slot leakage L_{al}, each representing flux paths that do not directly involve the rotor. Each of the components adds to the leakage inductance. The most prominent components of stator leakage are referred to as *slot, belt, zigzag, end winding, and skew*. Each of these will be discussed in the following paragraphs.

Belt and zigzag leakage components are due to air- gap space harmonics. As it turns out, these are relatively complicated to estimate, but we may get some notion from our first- order view of the machine. The trouble with estimating these leakage components is that they are not *really* independent of the rotor, even though we call them "leakage". *Belt* harmonics are of order $n = 5$ and $n = 7$. If there were no rotor coupling, the belt harmonic leakage terms would be:

$$X_{ag5} = \frac{3}{2}\frac{4}{\pi}\frac{\mu_0 N_S^2 k_5^2 Rl}{5^2 p^2 g} \tag{103}$$

$$X_{ag7} = \frac{3}{2}\frac{4}{\pi}\frac{\mu_0 N_S^2 k_7^2 Rl}{7^2 p^2 g} \tag{104}$$

The belt harmonics link to the rotor, however, and actually appear to be in parallel with components of rotor impedance appropriate to $5p$ and $7p$ pole- pair machines. At these harmonic orders we can usually ignore rotor resistance so that rotor impedance is purely inductive. Those components are:

$$X_{2,5} = \omega\frac{12l N_S^2 k_5^2}{N_R}L_{\text{slot}} + \omega\frac{6}{\pi}\frac{\mu_0 Rl N_S^2 k_5^2}{g}\left(\frac{1}{(N_R + 5p)^2} + \frac{1}{(N_R - 5p)^2}\right) \tag{105}$$

$$X_{2,7} = \omega\frac{12l N_S^2 k_7^2}{N_R}L_{\text{slot}} + \omega\frac{6}{\pi}\frac{\mu_0 Rl N_S^2 k_7^2}{g}\left(\frac{1}{(N_R + 7p)^2} + \frac{1}{(N_R - 7p)^2}\right) \tag{106}$$

In the simple model of the squirrel cage machine, because the rotor resistances are relatively small and slip high, the effect of rotor resistance is usually ignored. Then the fifth and seventh harmonic components of belt leakage are:

$$X_5 = X_{ag5}\|X_{2,5} \tag{107}$$

$$X_7 = X_{ag7}\|X_{2,7} \tag{108}$$

Stator zigzag leakage is from those harmonics of the orders $pn_s = N_{slots} \pm p$ where N_{slots}.

$$X_z = \frac{3}{2}\frac{4}{\pi}\frac{\mu_0 N_S^2 Rl}{g}\left(\frac{k_{n_s+}}{(N_{slots} + p)^2} + \frac{k_{n_s-}}{(N_{slots} - p)^2}\right) \tag{109}$$

Note that these harmonic orders do not tend to be shorted out by the rotor cage and so no direct interaction with the cage is ordinarily accounted for.

In order to reduce saliency effects that occur because the rotor teeth will tend to try to align with the stator teeth, induction motor designers always use a different number of slots in the rotor and stator. There still may be some tendency to align, and this produces "cogging" torques which in turn produce vibration and noise and, in severe cases, can retard or even prevent starting. To reduce this tendency to "cog", rotors are often built with a little "skew", or twist of the slots from one end to the other. Thus, when one tooth is aligned at one end of the machine, it is un-aligned at the other end. A side effect of this is to reduce the stator and rotor coupling by just a little, and this produces leakage reactance. This is fairly easy to estimate. Consider, for example, a space-fundamental flux density $B_r = B_1 \cos p\theta$, linking a (possibly) skewed full-pitch current path:

$$\lambda = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{\pi}{2p}+\frac{\varsigma}{p}\frac{x}{l}}^{\frac{\pi}{2p}+\frac{\varsigma}{p}\frac{x}{l}} B_1 \cos p\theta R d\theta dx$$

Here, the skew in the rotor is ς *electrical* radians from one end of the machine to the other. Evaluation of this yields:

$$\lambda = \frac{2 B_1 R l}{p} \frac{\sin \frac{\varsigma}{2}}{\frac{\varsigma}{2}}$$

Now, the difference between what would have been linked by a non-skewed rotor and what is linked by the skewed rotor is the skew leakage flux, now expressible as:

$$X_k = X_{ag} \left(1 - \frac{\sin \frac{\varsigma}{2}}{\frac{\varsigma}{2}} \right)$$

The final component of leakage reactance is due to the end windings. This is perhaps the most difficult of the machine parameters to estimate, being essentially three-dimensional in nature. There are a number of ways of estimating this parameter, but for our purposes we will use a simplified parameter from Alger[1]:

$$X_e = \frac{14}{4\pi^2} \frac{q}{2} \frac{\mu_0 R N_a^2}{p^2} (p - 0.3)$$

As with all such formulae, extreme care is required here, since we can give little guidance as to when this expression is correct or even close. And we will admit that a more complete treatment of this element of machine parameter construction would be an improvement.

4.1 Harmonic Order Rotor Resistance and Stray Load Losses

It is important to recognize that the machine rotor "sees" each of the stator harmonics in essentially the same way, and it is quite straightforward to estimate rotor parameters for the harmonic orders, as we have done just above. Now, particularly for the "belt" harmonic orders, there are rotor currents flowing in response to stator mmf's at fifth and seventh space harmonic order. The resistances attributable to these harmonic orders are:

$$R_{2,5} = \frac{12 l N_s^2 k_5^2}{N_R} R_{\text{slot},5} \tag{110}$$

$$R_{2,7} = \frac{12lN_s^2k_7^2}{N_R}R_{\text{slot},7} \tag{111}$$

The higher-order slot harmonics will have relative frequencies (slips) that are:

$$s_n = 1 \mp (1-s)n \left\{ \begin{array}{l} n = 6k+1 \\ n = 6k-1 \end{array} \right\} \text{k an integer} \tag{112}$$

The induction motor electromagnetic interaction can now be described by an augmented magnetic circuit as shown in Figure 17. Note that the terminal flux of the machine is the sum of *all* of the harmonic fluxes, and each space harmonic is excited by the same current so the individual harmonic components are in series.

Each of the space harmonics will have an electromagnetic interaction similar to the fundamental: power transferred across the air-gap is:

$$P_{em,n} = 3I_{2,n}^2 \frac{R_{2,n}}{s_n}$$

Of course dissipation in each circuit is:

$$P_{d,n} = 3I_{2,n}^2 R_{2,n}$$

leaving

$$P_{m,n} = 3I_{2,n}^2 \frac{R_{2,n}}{s_n}(1-s_n)$$

Note that this equivalent circuit has provision for two sets of circuits which look like "cages". In fact one of these sets is for the solid rotor body if that exists. We will discuss that anon. There is also a provision (r_c) for loss in the stator core iron.

Power deposited in the rotor harmonic resistance elements is characterized as "stray load" loss because it is not easily computed from the simple machine equivalent circuit.

4.2 Slot Models

Some of the more interesting things that can be done with induction motors have to do with the shaping of rotor slots to achieve particular frequency-dependent effects. We will consider here three cases, but there are many other possibilities.

First, suppose the rotor slots are representable as being rectangular, as shown in Figure 5, and assume that the slot dimensions are such that diffusion effects are not important so that current in the slot conductor is approximately uniform. In that case, the slot resistance and inductance per unit length are:

$$R_{\text{slot}} = \frac{1}{w_s h_s \sigma} \tag{113}$$

$$L_{\text{slot}} = \mu_0 \frac{h_s}{3w_s} \tag{114}$$

The slot resistance is obvious, the slot inductance may be estimated by recognizing that if the current in the slot is uniform, magnetic field crossing the slot must be:

$$H_y = \frac{I}{w_s}\frac{x}{h_s}$$

Then energy stored in the field in the slot is simply:

$$\frac{1}{2}L_{\text{slot}}I^2 = w_s \int_0^{h_s} \frac{\mu_0}{2}\left(\frac{Ix}{w_s h_s}\right)^2 dx = \frac{1}{6}\frac{\mu_0 h_s}{w_s}I^2$$
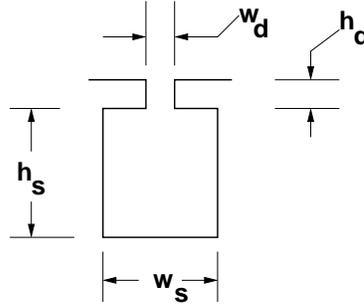

Figure 5: Single Slot

4.3 Deep Slots

Now, suppose the slot is *not* small enough that diffusion effects can be ignored. The slot becomes "deep" to the extent that its depth is less than (or even comparable to) the *skin depth* for conduction at slip frequency. Conduction in this case may be represented by using the Diffusion Equation:

$$\nabla^2 \overline{H} = \mu_0 \sigma \frac{\partial \overline{H}}{\partial t}$$

In the steady state, and assuming that only cross-slot flux (in the y direction) is important, and the only variation that is important is in the radial (x) direction:

$$\frac{\partial^2 H_y}{\partial x^2} = j\omega_s \mu_0 \sigma H_y$$

This is solved by solutions of the form:

$$H_y = H_{\pm}e^{\pm(1+j)\frac{x}{\delta}}$$

where the skin depth is

$$\delta = \sqrt{\frac{2}{\omega_s \mu_0 \sigma}}$$

Since H_y must vanish at the bottom of the slot, it must take the form:

$$H_y = H_{\text{top}}\frac{\sinh(1+j)\frac{x}{\delta}}{\sinh(1+j)\frac{h_s}{\delta}}$$

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Since current is the curl of magnetic field,

$$J_z = \sigma E_z = \frac{\partial H_y}{\partial x} = H_{\text{top}} \frac{1+j}{\delta} \frac{\cosh(1+j)\frac{h_s}{\delta}}{\sinh(1+j)\frac{h_s}{\delta}}$$

Then slot impedance, per unit length, is:

$$Z_{\text{slot}} = \frac{1}{w_s} \frac{1+j}{\sigma\delta} \coth(1+j)\frac{h_s}{\delta}$$

Of course the impedance (purely reactive) due to the slot depression must be added to this. It is possible to extract the real and imaginary parts of this impedance (the process is algebraically a bit messy) to yield:

$$R_{\text{slot}} = \frac{1}{w_s\sigma\delta} \frac{\sinh 2\frac{h_s}{\delta} + \sin 2\frac{h_s}{\delta}}{\cosh 2\frac{h_s}{\delta} - \cos 2\frac{h_s}{\delta}}$$

$$L_{\text{slot}} = \mu_0 \frac{h_d}{w_d} + \frac{1}{\omega_s} \frac{1}{w_s\sigma\delta} \frac{\sinh 2\frac{h_s}{\delta} - \sin 2\frac{h_s}{\delta}}{\cosh 2\frac{h_s}{\delta} - \cos 2\frac{h_s}{\delta}}$$

4.4   Multiple Cages

The purpose of a "deep" slot is to improve starting performance of a motor. When the rotor is stationary, the frequency seen by rotor conductors is relatively high, and current crowding due to the skin effect makes rotor resistance appear to be high. As the rotor accelerates the frequency seen from the rotor drops, lessening the skin effect and making more use of the rotor conductor. This, then, gives the machine higher starting torque (requiring high resistance) without compromising running efficiency.

This effect can be carried even further by making use of *multiple cages*, such as is shown in Figure 6. Here there are two conductors in a fairly complex slot. Estimating the impedance of this slot is done in stages to build up an equivalent circuit.

Assume for the purposes of this derivation that each section of the multiple cage is small enough that currents can be considered to be uniform in each conductor. Then the bottom section may be represented as a resistance in series with an inductance:

$$R_a = \frac{1}{\sigma w_1 h_1}$$

$$L_a = \frac{\mu_0}{3} \frac{h_1}{w_1}$$

The narrow slot section with no conductor between the top and bottom conductors will contribute an inductive impedance:

$$L_s = \mu_0 \frac{h_s}{w_s}$$

The top conductor will have a resistance:
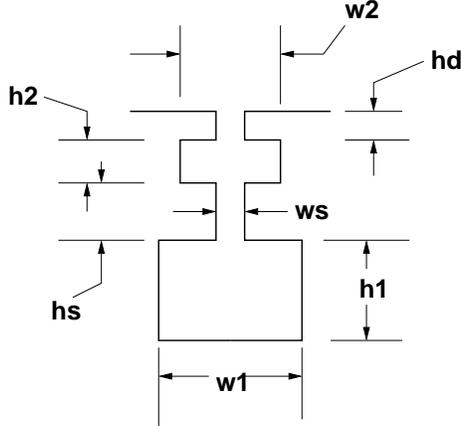
$$R_b = \frac{1}{\sigma w_2 h_2}$$

Figure 6: Double Slot

Now, in the equivalent circuit, current flowing in the lower conductor will produce a magnetic field across this section, yielding a series inductance of

$$L_b = \mu_0 \frac{h_2}{w_2}$$

By analogy with the bottom conductor, current in the top conductor flows through only one third of the inductance of the top section, leading to the equivalent circuit of Figure 7, once the inductance of the slot depression is added on:
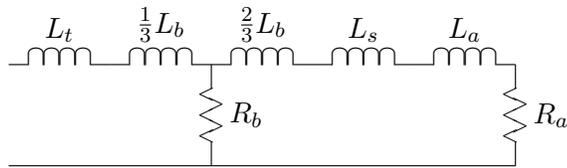
$$L_t = \mu_0 \frac{h_d}{w_d}$$

Figure 7: Equivalent Circuit: Double Bar

Now, this rotor bar circuit fits right into the framework of the induction motor equivalent circuit, shown for the double cage case in Figure 8, with

$$R_{2a} = \frac{12 l N_S^2 k_S^2}{N_R} R_a$$

$$R_{2b} = \frac{12 l N_S^2 k_S^2}{N_R} R_b$$

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$$
\begin{aligned}
X_{2a} &= \omega \frac{12 l N_S^2 k_S^2}{N_R} (\frac{2}{3} L_b + L_s + L_a) \\
X_{2a} &= \omega \frac{12 l N_S^2 k_S^2}{N_R} (L_t + \frac{1}{3} L_b)
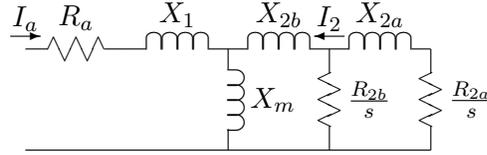\end{aligned}
$$

Figure 8: Equivalent Circuit: Double Cage Rotor

4.5 Rotor End Ring Effects

It is necessary to correct for "end ring" resistance in the rotor. To do this, we note that the magnitude of surface current density in the rotor is related to the magnitude of individual bar current by:

$$
I_z = K_z \frac{2\pi R}{N_R} \tag{115}
$$

Current in the end ring is:

$$
I_R = K_z \frac{R}{p} \tag{116}
$$

Then it is straightforward to calculate the ratio between power dissipated in the end rings to power dissipated in the conductor bars themselves, considering the ratio of current densities and volumes. Assuming that the bars and end rings have the same *radial* extent, the ratio of current densities is:

$$
\frac{J_R}{J_z} = \frac{N_R}{2\pi p} \frac{w_r}{l_r} \tag{117}
$$

where w_r is the average width of a conductor bar and l_r is the axial end ring length.

Now, the ratio of losses (and hence the ratio of resistances) is found by multiplying the square of current density ratio by the ratio of volumes. This is approximately:

$$
\frac{R_{\text{end}}}{R_{\text{slot}}} = \left(\frac{N_R}{2\pi p} \frac{w_r}{l_r} \right)^2 2 \frac{2\pi R}{N_R l} \frac{l_r}{w_r} = \frac{N_R R w_r}{\pi l l_r p^2} \tag{118}
$$

4.6 Windage

Bearing friction, windage loss and fan input power are often regarded as elements of a "black art". We approach them with some level of trepidation, for motor manufacturers seem to take a highly

empirical view of these elements. What follows is an attempt to build reasonable but simple models for two effects: loss in the air gap due to windage and input power to the fan for cooling. Some caution is required here, for these elements of calculation have *not* been properly tested, although they seem to give reasonable numbers

The first element is gap windage loss. This is produced by shearing of the air in the relative rotation gap. It is likely to be a signifigant element only in machines with very narrow air gaps or very high surface speeds. But these include, of course, the high performance machines with which we are most interested. We approach this with a simple "couette flow" model. Air-gap shear loss is approximately:

$$P_w = 2\pi R^4 \Omega^3 l \rho_a f \tag{119}$$

where ρ_a is the density of the air-gap medium (possibly air) and f is the *friction factor*, estimated by:

$$f = \frac{.0076}{R_n^{\frac{1}{4}}} \tag{120}$$

and the *Reynold's Number* R_n is

$$R_n = \frac{\Omega R g}{\nu_{\text{air}}} \tag{121}$$

and ν_{air} is the kinematic viscosity of the air-gap medium.

The second element is fan input power. We base an estimate of this on two hypotheses. The first of these is that the mass flow of air circulated by the fan can be calculated by the loss in the motor and an average temperature rise in the cooling air. The second hypothesis is the the pressure rise of the fan is established by the centrifugal pressure rise associated with the surface speed at the outside of the rotor. Taking these one at a time: If there is to be a temperature rise ΔT in the cooling air, then the mass flow volume is:

$$\dot{m} = \frac{P_d}{C_p \Delta T}$$

and then volume flow is just

$$\dot{v} = \frac{\dot{m}}{\rho_{\text{air}}}$$

Pressure rise is estimated by centrifugal force:

$$\Delta P = \rho_{\text{air}} \left(\frac{\omega}{p} r_{\text{fan}} \right)^2$$

then power is given by:

$$P_{\text{fan}} = \Delta P \dot{v}$$

For reference, the properties of air are:

Density	ρ_{air}	1.18	kg/m^2
Kinematic Viscosity	ν_{air}	1.56×10^{-5}	m^2/sec
Heat Capacity	C_p	1005.7	J/kg

4.7 Magnetic Circuit Loss and Excitation

There will be some loss in the stator magnetic circuit due to eddy current and hysteresis effects in the core iron. In addition, particularly if the rotor and stator teeth are saturated there will be MMF expended to push flux through those regions. These effects are very difficult to estimate from first principles, so we resort to a simple model.

Assume that the loss in saturated steel follows a law such as:

$$P_d = P_B \left(\frac{\omega_e}{\omega_B}\right)^{\epsilon_f} \left(\frac{B}{B_B}\right)^{\epsilon_b} \tag{122}$$

This is not *too* bad an estimate for the behavior of core iron. Typically, ϵ_f is a bit less than two (between about 1.3 and 1.6) and ϵ_b is a bit more than two (between about 2.1 and 2.4). Of course this model is good only for a fairly restricted range of flux density. Base dissipation is usually expressed in "watts per kilogram", so we first compute flux density and then mass of the two principal components of the stator iron, the teeth and the back iron.

In a similar way we can model the exciting volt-amperes consumed by core iron by something like:

$$Q_c = \left(V a_1 \left(\frac{B}{B_B}\right)^{\epsilon_{v1}} + V a_2 \left(\frac{B}{B_B}\right)^{\epsilon_{v2}}\right) \frac{\omega}{\omega_B} \tag{123}$$

This, too, is a form that appears to be valid for some steels. Quite obviously it may be necessary to develop different forms of curve 'fits' for different materials.

Flux density (RMS) in the air-gap is:

$$B_r = \frac{pV_a}{2RlN_a k_1 \omega_s} \tag{124}$$

Then flux density in the stator teeth is:

$$B_t = B_r \frac{w_t + w_1}{w_t} \tag{125}$$

where w_t is tooth width and w_1 is slot top width. Flux in the back-iron of the core is

$$B_c = B_r \frac{R}{pd_c} \tag{126}$$

where d_c is the radial depth of the core.

One way of handling this loss is to assume that the core handles flux corresponding to terminal voltage, add up the losses and then compute an equivalent resistance and reactance:

$$r_c = \frac{3|V_a|^2}{P_{\text{core}}}$$

$$x_c = \frac{3|V_a|^2}{Q_{\text{core}}}$$

then put this equivalent resistance in parallel with the air-gap reactance element in the equivalent circuit.

5 Solid Iron Rotor Bodies

Solid steel rotor electric machines (SSRM) can be made to operate with very high surface speeds and are thus suitable for use in high RPM situations. They resemble, in form and function, hysteresis machines. However, asynchronous operation will produce higher power output because it takes advantage of higher flux density. We consider here the interactions to be expected from solid iron rotor bodies. The equivalent circuits can be placed in parallel (harmonic-by-harmonic) with the equivalent circuits for the squirrel cage, if there is also a cage in the machine.

To estimate the rotor parameters R_{2s} and X_{2s}, we assume that important field quantities in the machine are sinusoidally distributed in time and space, so that radial flux density is:

$$B_r = Re\left(\underline{B}_r e^{j(\omega t - p\phi)}\right) \tag{127}$$

and, similarly, axially directed rotor surface current is:

$$K_z = Re\left(\underline{K}_z e^{j(\omega t - p\phi)}\right) \tag{128}$$

Now, since by Faraday's law:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{129}$$

we have, in this machine geometry:

$$\frac{1}{R}\frac{\partial}{\partial \phi}E_z = -\frac{\partial B_r}{\partial t} \tag{130}$$

The transformation between rotor and stator coordinates is:

$$\phi' = \phi - \omega_m t \tag{131}$$

where ω_m is rotor speed. Then:

$$p\omega_m = \omega - \omega_r = \omega(1 - s) \tag{132}$$

and

Now, axial electric field is, in the frame of the rotor, just:

$$E_z = Re\left(\underline{E}_z e^{j(\omega t - p\phi)}\right) \tag{133}$$

$$= Re\left(\underline{E}_z e^{j(\omega_r t - p\phi')}\right) \tag{134}$$

and

$$\underline{E}_z = \frac{\omega_r R}{p}\underline{B}_r \tag{135}$$

Of course electric field in the rotor frame is related to rotor surface current by:

$$\underline{E}_z = \underline{Z}_s \underline{K}_z \tag{136}$$

Now these quantities can be related to the stator by noting that *air-gap* voltage is related to radial flux density by:

$$\underline{B}_r = \frac{p}{2lN_a k_1 R\omega}V_{ag} \tag{137}$$

The stator-equivalent rotor current is:

$$\underline{I}_2 = \frac{\pi}{3} \frac{R}{N_a k_a} \underline{K}_z \tag{138}$$

Then we can find stator referred, rotor equivalent impedance to be:

$$\underline{Z}_2 = \frac{\underline{V}_{ag}}{\underline{I}_2} = \frac{3}{2} \frac{4}{\pi} \frac{l}{R} N_a^2 k_a^2 \frac{\omega}{\omega_r} \frac{E_z}{\underline{K}_z} \tag{139}$$

Now, if rotor surface impedance can be expressed as:

$$\underline{Z}_s = R_s + j\omega_r L_s \tag{140}$$

then

$$\underline{Z}_2 = \frac{R_2}{s} + jX_2 \tag{141}$$

where

$$R_2 = \frac{3}{2} \frac{4}{\pi} \frac{l}{R} N_a^2 k_1^2 R_s \tag{142}$$

$$X_2 = \frac{3}{2} \frac{4}{\pi} \frac{l}{R} N_a^2 k_1^2 X_s \tag{143}$$

Now, to find the rotor surface impedance, we make use of a nonlinear eddy-current model proposed by Agarwal. First we define an equivalent penetration depth (similar to a skin depth):

$$\delta = \sqrt{\frac{2H_m}{\omega_r \sigma B_0}} \tag{144}$$

where σ is rotor surface material volume conductivity, B_0, "saturation flux density" is taken to be 75 % of actual saturation flux density and

$$H_m = |\underline{K}_z| = \frac{3}{\pi} \frac{N_a k_a}{R} |\underline{I}_2| \tag{145}$$

Then rotor surface resistivity and surface reactance are:

$$R_s = \frac{16}{3\pi} \frac{1}{\sigma \delta} \tag{146}$$

$$X_s = .5R_s \tag{147}$$

Note that the rotor elements X_2 and R_2 depend on rotor current I_2, so the problem is nonlinear. We find, however, that a simple iterative solution can be used. First we make a guess for R_2 and find currents. Then we use those currents to calculate R_2 and solve again for current. This procedure is repeated until convergence, and the problem seems to converge within just a few steps.

Aside from the necessity to iterate to find rotor elements, standard network techniques can be used to find currents, power input to the motor and power output from the motor, torque, etc.

5.1 Solution

Not all of the equivalent circuit elements are known as we start the solution. To start, we assume a value for R_2, possibly some fraction of X_m, but the value chosen doesn not seem to matter much. The rotor reactance X_2 is just a fraction of R_2. Then, we proceed to compute an "air-gap" impedance, just the impedance looking into the parallel combination of magnetizing and rotor branches:

$$Z_g = jX_m || (jX_2 + \frac{R_2}{s}) \tag{148}$$

(Note that, for a generator, slip s is negative).

A total impedance is then

$$Z_t = jX_1 + R_1 + Z_g \tag{149}$$

and terminal current is

$$I_t = \frac{V_t}{Z_t} \tag{150}$$

Rotor current is just:

$$I_2 = I_t \frac{jX_m}{jX_2 + \frac{R_2}{s}} \tag{151}$$

Now it is necessary to iteratively correct rotor impedance. This is done by estimating flux density at the surface of the rotor using (145), then getting a rotor surface impedance using (146) and using *that* and (143 to estimate a new value for R_2. Then we start again with (148). The process "drops through" this point when the new and old estimates for R_2 agree to some criterion.

5.2 Harmonic Losses in Solid Steel

If the rotor of the machine is constructed of solid steel, there will be eddy currents induced on the rotor surface by the higher-order space harmonics of stator current. These will produce magnetic fields and losses. This calculation assumes the rotor surface is linear and smooth and can be characterized by a conductivity and relative permeability. In this discussion we include two space harmonics (positive and negative going). In practice it may be necessary to carry four (or even more) harmonics, including both 'belt' and 'zigzag' order harmonics.

Terminal current produces magnetic field in the air-gap for each of the space harmonic orders, and each of these magnetic fields induces rotor currents of the same harmonic order.

The "magnetizing" reactances for the two harmonic orders, really the two components of the zigzag leakage, are:

$$X_{zp} = X_m \frac{k_p^2}{N_p^2 k_1^2} \tag{152}$$

$$X_{zn} = X_m \frac{k_n^2}{N_n^2 k_1^2} \tag{153}$$

where N_p and N_n are the positive and negative going harmonic orders: For 'belt' harmonics these orders are 7 and 5. For 'zigzag' they are:

$$N_p = \frac{N_s + p}{p} \tag{154}$$

$$N_n = \frac{N_s - p}{p} \tag{155}$$

Now, there will be a current on the surface of the rotor at each harmonic order, and following 67, the equivalent rotor element current is:

$$\underline{I}_{2p} = \frac{\pi}{3} \frac{R}{N_a k_p} \underline{K}_p \tag{156}$$

$$\underline{I}_{2n} = \frac{\pi}{3} \frac{R}{N_a k_n} \underline{K}_n \tag{157}$$

These currents flow in response to the magnetic field in the air-gap which in turn produces an axial electric field. Viewed from the rotor this electric field is:

$$\underline{E}_p = s_p \omega R \underline{B}_p \tag{158}$$

$$\underline{E}_n = s_n \omega R \underline{B}_n \tag{159}$$

where the *slip* for each of the harmonic orders is:

$$s_p = 1 - N_p(1 - s) \tag{160}$$

$$s_n = 1 + N_p(1 - s) \tag{161}$$

and then the surface currents that flow in the surface of the rotor are:

$$\underline{K}_p = \frac{\underline{E}_p}{Z_{sp}} \tag{162}$$

$$\underline{K}_n = \frac{\underline{E}_n}{Z_{sn}} \tag{163}$$

where Z_{sp} and Z_{sn} are the *surface* impedances at positive and negative harmonic slip frequencies, respectively. Assuming a linear surface, these are, approximately:

$$Z_s = \frac{1 + j}{\sigma \delta} \tag{164}$$

where σ is material restivity and the skin depth is

$$\delta = \sqrt{\frac{2}{\omega_s \mu \sigma}} \tag{165}$$

and ω_s is the frequency of the given harmonic from the rotor surface. We can postulate that the appropriate value of μ to use is the same as that estimated in the *nonlinear* calculation of the space fundamental, but this requires empirical confirmation.

The voltage induced in the stator by each of these space harmonic magnetic fluxes is:

$$V_p = \frac{2 N_a k_p l R \omega}{N_p p} \underline{B}_p \tag{166}$$

$$V_n = \frac{2 N_a k_n l R \omega}{N_n p} \underline{B}_n \tag{167}$$

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Then the equivalent circuit impedance of the rotor is just:

$$Z_{2p} = \frac{V_p}{I_p} = \frac{3}{2}\frac{4}{\pi}\frac{N_a^2 k_p^2 l}{N_p R}\frac{Z_{sp}}{s_p} \tag{168}$$

$$Z_{2n} = \frac{V_n}{I_n} = \frac{3}{2}\frac{4}{\pi}\frac{N_a^2 k_n^2 l}{N_n R}\frac{Z_{sn}}{s_n} \tag{169}$$

The equivalent rotor circuit elements are now:

$$R_{2p} = \frac{3}{2}\frac{4}{\pi}\frac{N_a^2 k_p^2 l}{N_p R}\frac{1}{\sigma\delta_p} \tag{170}$$

$$R_{2n} = \frac{3}{2}\frac{4}{\pi}\frac{N_a^2 k_n^2 l}{N_n R}\frac{1}{\sigma\delta_n} \tag{171}$$

$$X_{2p} = \frac{1}{2}R_{2p} \tag{172}$$

$$X_{2n} = \frac{1}{2}R_{2n} \tag{173}$$

5.3 Stray Losses

So far in this document, we have outlined the major elements of torque production and consequently of machine performance. We have also discussed, in some cases, briefly, the major sources of loss in induction machines. Using what has been outlined in this document will give a reasonable impression of how an induction machine works. We have also discussed some of the *stray load* losses: those which can be (relatively) easily accounted for in an equivalent circuit description of the machine. But there are other losses which will occur and which are harder to estimate. We do not claim to do a particularly accurate job of estimating these losses, and fortunately they do not normally turn out to be very large. To be accounted for here are:

1. No-load losses in rotor teeth because of stator slot opening modulation of fundamental flux density,

2. Load losses in the rotor teeth because of stator zigzag mmf, and

3. No-load losses in the solid rotor body (if it exists) due to stator slot opening modulation of fundamental flux density.

Note that these losses have a somewhat different character from the other miscellaneous losses we compute. They show up as *drag* on the rotor, so we subtract their power from the mechanical output of the machine. The first and third of these are, of course, very closely related so we take them first.

The stator slot openings 'modulate' the space fundamental magnetic flux density. We may estimate a slot opening angle (relative to the slot pitch):

$$\theta_D = \frac{2\pi w_d N_s}{2\pi r} = \frac{w_d N_s}{r}$$

Then the amplitude of the magnetic field disturbance is:

$$B_H = B_{r1}\frac{2}{\pi}\sin\frac{\theta_D}{2}$$

In fact, this flux disturbance is really in the form of two traveling waves, one going forward and one backward with respect to the stator at a velocity of ω/N_s. Since operating slip is relatively small, the two variations will have just about the same frequency as viewed from the rotor, so it seems reasonable to lump them together. The frequency is:

$$\omega_H = \omega\frac{N_s}{p}$$

Now, for laminated rotors this magnetic field modulation will affect the tips of rotor teeth. We assume (perhaps arbitrarily) that the loss due to this magnetic field modulation can be estimated from ordinary steel data (as we estimated core loss above) and that only the rotor teeth, not any of the rotor body, are affected. The method to be used is straightforward and follows almost exactly what was done for core loss, with modification only of the frequency and field amplitude.

For solid steel rotors the story is only a little different. The magnetic field will produce an axial electric field:

$$\underline{E}_z = R\frac{\omega}{p}B_H$$

and that, in turn, will drive a surface current

$$\underline{K}_z = \frac{\underline{E}_z}{\underline{Z}_s}$$

Now, what is important is the magnitude of the surface current, and since $|\underline{Z}_s| = \sqrt{1 + .5^2}R_s \approx 1.118R_s$, we can simply use rotor resistance. The nonlinear surface penetration depth is:

$$\delta = \sqrt{\frac{2B_0}{\omega_H\sigma|\underline{K}_z|}}$$

A brief iterative substitution, re-calculating δ and then $|\underline{K}_z|$ quickly yields consistent values for δ and R_s. Then the full-voltage dissipation is:

$$P_{rs} = 2\pi Rl\frac{|\underline{K}_z|^2}{\sigma\delta}$$

and an equivalent resistance is:

$$R_{rs} = \frac{3|V_a|^2}{P_{rs}}$$

Finally, the zigzag order current harmonics in the stator will produce magnetic fields in the air gap which will drive magnetic losses in the teeth of the rotor. Note that this is a bit different from the modulation of the space fundamental produced by the stator slot openings (although the harmonic order will be the same, the spatial orientation will be different and will vary with load current). The magnetic flux in the air-gap is most easily related to the equivalent circuit voltage on the nth harmonic:

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$$B_n = \frac{npv_n}{2lRN_ak_nomega}$$

This magnetic field variation will be substantial only for the zigzag order harmonics: the belt harmonics will be essentially shorted out by the rotor cage and those losses calculated within the equivalent circuit. The frequency seen by the rotor is that of the space harmonics, already calculated, and the loss can be estimated in the same way as core loss, although as we have pointed out it appears as a 'drag' on the rotor.

6 Induction Motor Speed Control

6.1 Introduction

The inherent attributes of induction machines make them very attractive for drive applications. They are rugged, economical to build and have no sliding contacts to wear. The difficulty with using induction machines in servomechanisms and variable speed drives is that they are "hard to control", since their torque-speed relationship is complex and nonlinear. With, however, modern power electronics to serve as frequency changers and digital electronics to do the required arithmetic, induction machines are seeing increasing use in drive applications.

In this chapter we develop models for control of induction motors. The derivation is quite brief for it relies on what we have already done for synchronous machines. In this chapter, however, we will stay in "ordinary" variables, skipping the per-unit normalization.

6.2 Volts/Hz Control

Remembering that induction machines generally tend to operate at relatively low *per unit* slip, we might conclude that one way of building an adjustable speed drive would be to supply an induction motor with adjustable stator frequency. And this is, indeed, possible. One thing to remember is that flux is inversely proportional to frequency, so that to maintain constant flux one must make stator voltage proportional to frequency (hence the name "constant volts/Hz"). However, voltage supplies are always limited, so that at some frequency it is necessary to switch to constant voltage control. The analogy to DC machines is fairly direct here: below some "base" speed, the machine is controlled in constant flux ("volts/Hz") mode, while above the base speed, flux is inversely proportional to speed. It is easy to see that the maximum torque is then inversely to the square of flux, or therefore to the square of frequency.

To get a first-order picture of how an induction machine works at adjustable speed, start with the simplified equivalent network that describes the machine, as shown in Figure 9

Earlier in this chapter, it was shown that torque can be calculated by finding the power dissipated in the virtual resistance R_2/s and dividing by electrical speed. For a three phase machine, and assuming we are dealing with RMS magnitudes:

$$T_e = 3\frac{p}{\omega}|I_2|^2\frac{R_2}{s}$$

where ω is the electrical frequency and p is the number of pole pairs. It is straightforward to find I_2 using network techniques. As an example, Figure 10 shows a series of torque/speed curves for
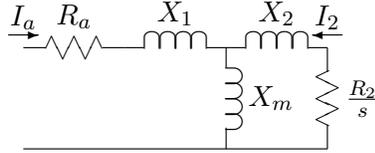
Figure 9: Equivalent Circuit

an induction machine operated with a wide range of input frequencies, both below and above its "base" frequency. The parameters of this machine are:

Number of Phases	3
Number of Pole Pairs	3
RMS Terminal Voltage (line-line)	230
Frequency (Hz)	60
Stator Resistance R_1	.06 Ω
Rotor Resistance R_2	.055 Ω
Stator Leakage X_1	.34 Ω
Rotor Leakage X_2	.33 Ω
Magnetizing Reactance X_m	10.6 Ω

Strategy for operating the machine is to make terminal voltage magnitude proportional to frequency for input frequencies less than the "Base Frequency", in this case 60 Hz, and to hold voltage constant for frequencies above the "Base Frequency".

For high frequencies the torque production falls fairly rapidly with frequency (as it turns out, it is roughly proportional to the inverse of the square of frequency). It also falls with very low frequency because of the effects of terminal resistance. We will look at this next.

6.3   Idealized Model: No Stator Resistance

Ignore, for the moment, R_1. An equivalent circuit is shown in Figure 11. It is fairly easy to show that, from the rotor, the combination of source, armature leakage and magnetizing branch can be replaced by its equivalent circuit, as shown in in Figure 12.

In the circuit of Figure 12, the parameters are:

$$V' = V\frac{X_m}{X_m + X_1}$$
$$X' = X_m||X_1$$

If the machine is operated at variable frequency ω, but the reactance is established at frequency ω_B, current is:

$$\underline{I} = \frac{V'}{j(X_1' + X_2)\frac{\omega}{\omega_B} + \frac{R_2}{s}}$$
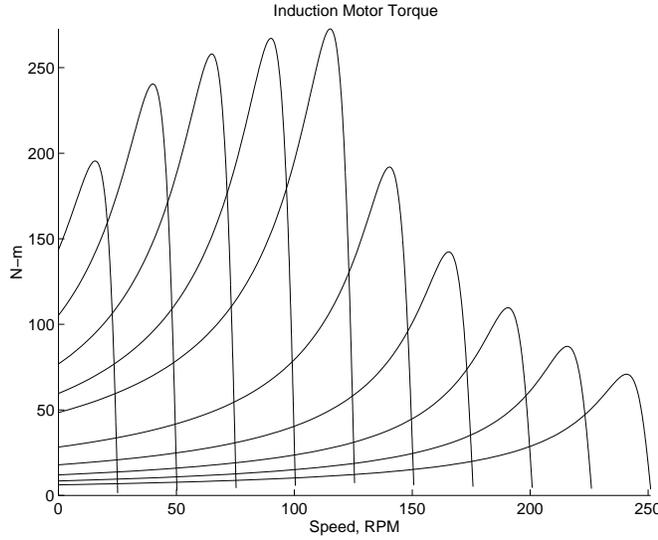
32

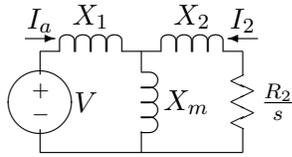Figure 10: Induction Machine Torque-Speed Curves

Figure 11: Idealized Circuit: Ignore Armature Resistance

and then torque is

$$T_e = 3|I_2|^2 \frac{R_2}{s} = \frac{3p}{\omega} \frac{|V'|^2 \frac{R_2}{s}}{(X_1' + X_2)^2 + (\frac{R_2}{s})^2}$$

Now, if we note that what counts is the *absolute* slip of the rotor, we might define a slip with respect to base frequency:

$$s = \frac{\omega_r}{\omega} = \frac{\omega_r}{\omega_B} \frac{\omega_B}{\omega} = s_B \frac{\omega_B}{\omega}$$

Then, if we assume that voltage is applied proportional to frequency:

$$V' = V_0' \frac{\omega}{\omega_B}$$

and with a little manipulation, we get:

$$T_e = \frac{3p}{\omega_B} \frac{|V_0'|^2 \frac{R_2}{s_B}}{(X_1' + X_2)^2 + (\frac{R_2}{s_B})^2}$$
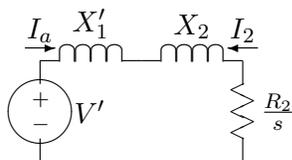
33

Figure 12: Idealized Equivalent

This would imply that torque is, if voltage is proportional to frequency, meaning constant applied flux, dependent only on absolute slip. The torque-speed curve is a constant, dependent only on the difference between synchronous and actual rotor speed.

This is fine, but eventually, the notion of "volts per Hz" runs out because at some number of Hz, there are no more volts to be had. This is generally taken to be the "base" speed for the drive. Above that speed, voltage is held constant, and torque is given by:

$$T_e = \frac{3p}{\omega_B} \frac{|V'|^2 \frac{R_2}{s_B}}{(X_1' + X_2)^2 + (\frac{R_2}{s_B})^2}$$

The peak of this torque has a square-inverse dependence on frequency, as can be seen from Figure 13.
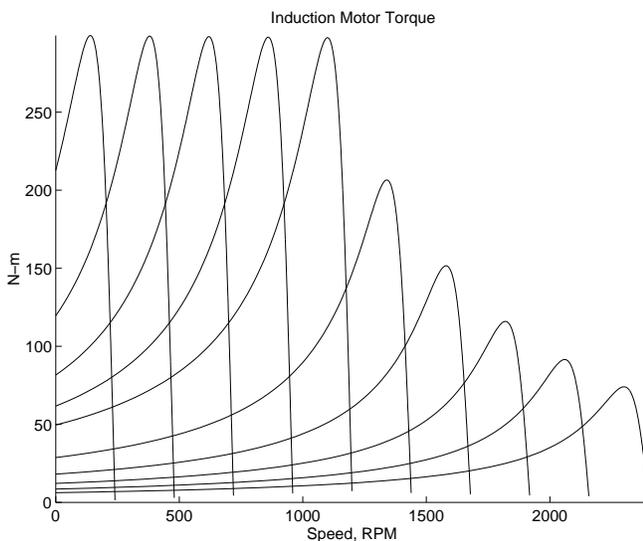

Figure 13: Idealized Torque-Speed Curves: Zero Stator Resistance

6.4 Peak Torque Capability

Assuming we have a smart controller, we are interested in the actual capability of the machine. At some voltage and frequency, torque is given by:

$$T_e = 3|I_2|^2 \frac{R_2}{s} = \frac{3\frac{p}{\omega}|V'|^2 \frac{R_2}{s}}{((X_1' + X_2)(\frac{\omega}{\omega_B}))^2 + (R_1' + \frac{R_2}{s})^2}$$

Now, we are interested in finding the *peak* value of that, which is given by the value of $R_2 s$ which maximizes power transfer to the virtual resistance. This is given by the matching condition:

$$\frac{R_2}{s} = \sqrt{R_1'^2 + ((X_1' + X_2)(\frac{\omega}{\omega_B}))^2}$$

Then maximum (breakdown) torque is given by:

$$T_{\max} = \frac{\frac{3p}{\omega}|V'|^2 \sqrt{R_1'^2 + ((X_1' + X_2)(\frac{\omega}{\omega_B}))^2}}{((X_1' + X_2)(\frac{\omega}{\omega_B}))^2 + (R_1' + \sqrt{R_1'^2 + ((X_1' + X_2)(\frac{\omega}{\omega_B}))^2})^2}$$

This is plotted in Figure 14. Just as a check, this was calculated assuming $R_1 = 0$, and the results are plotted in figure 15. This plot shows, as one would expect, a constant torque limit region to zero speed.
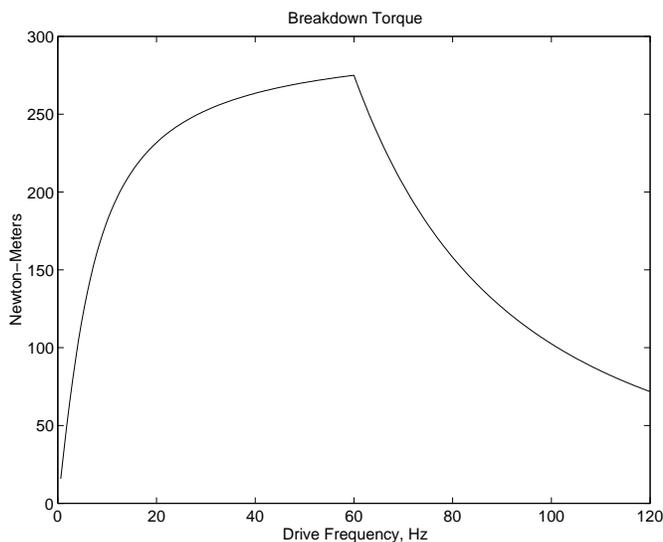

Figure 14: Torque-Capability Curve For An Induction Motor

6.5 Field Oriented Control

One of the more useful impacts of modern power electronics and control technology has enabled us to turn induction machines into high performance servomotors. In this note we will develop a
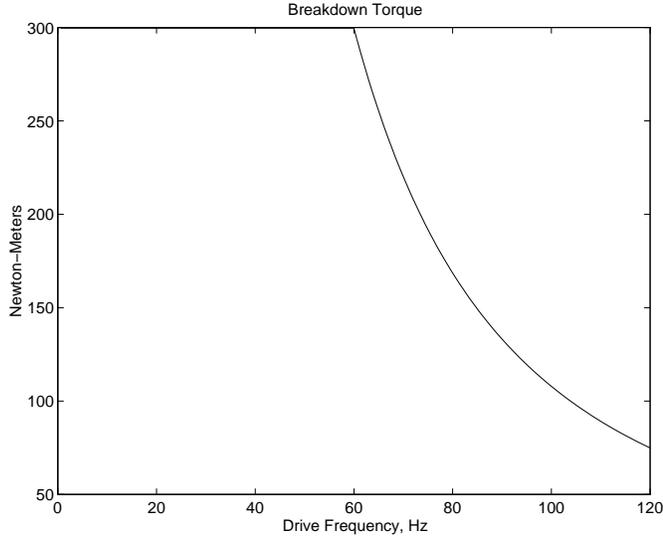
Figure 15: Idealized Torque Capability Curve: Zero Stator Resistance

picture of how this is done. Quite obviously there are many details which we will not touch here. The objective is to emulate the performance of a DC machine, in which (as you will recall), torque is a simple function of applied current. For a machine with one field winding, this is simply:

$$T = GI_f I_a$$

This makes control of such a machine quite easy, for once the desired torque is known it is easy to translate that torque command into a current and the motor does the rest.

Of course DC (commutator) machines are, at least in large sizes, expensive, not particularly efficient, have relatively high maintenance requirements because of the sliding brush/commutator interface, provide environmental problems because of sparking and carbon dust and are environmentally sensitive. The induction motor is simpler and more rugged. Until fairly recently the induction motor has not been widely used in servo applications because it was thought to be "hard to control". As we will show, it does take a little effort and even some computation to do the controls right, but this is becoming increasingly affordable.

6.6   Elementary Model:

We return to the elementary model of the induction motor. In ordinary variables, referred to the stator, the machine is described by flux-current relationships (in the d-q reference frame):

$$
\begin{bmatrix} \lambda_{dS} \\ \lambda_{dR} \end{bmatrix} = \begin{bmatrix} L_S & M \\ M & L_R \end{bmatrix} \begin{bmatrix} i_{dS} \\ i_{dR} \end{bmatrix}
$$
$$
\begin{bmatrix} \lambda_{qS} \\ \lambda_{qR} \end{bmatrix} = \begin{bmatrix} L_S & M \\ M & L_R \end{bmatrix} \begin{bmatrix} i_{qS} \\ i_{qR} \end{bmatrix}
$$

Note the machine is symmetric (there is no saliency), and since we are referred to the stator, the stator and rotor self-inductances include leakage terms:

$$
\begin{aligned}
L_S &= M + L_{S\ell} \\
L_R &= M + L_{R\ell}
\end{aligned}
$$

The voltage equations are:

$$
\begin{aligned}
v_{dS} &= \frac{d\lambda_{dS}}{dt} - \omega\lambda_{qS} + r_S i_{dS} \\
v_{qS} &= \frac{d\lambda_{qS}}{dt} + \omega\lambda_{dS} + r_S i_{qS} \\
0 &= \frac{d\lambda_{dR}}{dt} - \omega_s\lambda_{qR} + r_R i_{dR} \\
0 &= \frac{d\lambda_{qR}}{dt} + \omega_s\lambda_{dR} + r_R i_{qR}
\end{aligned}
$$

Note that both rotor and stator have "speed" voltage terms since they are both rotating with respect to the rotating coordinate system. The speed of the rotating coordinate system is w with respect to the stator. With respect to the rotor that speed is , where wm is the rotor mechanical speed. Note that this analysis does not require that the reference frame coordinate system speed w be constant.

Torque is given by:

$$
T^e = \frac{3}{2}p\left(\lambda_{dS}i_{qS} - \lambda_{qS}i_{dS}\right)
$$

6.7 Simulation Model

As a first step in developing a simulation model, see that the inversion of the flux-current relationship is (we use the d- axis since the q- axis is identical):

$$
\begin{aligned}
i_{dS} &= \frac{L_R}{L_S L_R - M^2}\lambda_{dS} - \frac{M}{L_S L_R - M^2}\lambda_{dR} \\
i_{dR} &= \frac{M}{L_S L_R - M^2}\lambda_{dS} - \frac{L_S}{L_S L_R - M^2}\lambda_{dR}
\end{aligned}
$$

Now, if we make the following definitions (the motivation for this should by now be obvious):

$$
\begin{aligned}
X_d &= \omega_0 L_S \\
X_{kd} &= \omega_0 L_R \\
X_{ad} &= \omega_0 M \\
X'_d &= \omega_0\left(L_S - \frac{M^2}{L_R}\right)
\end{aligned}
$$

the currents become:

$$
\begin{aligned}
i_{dS} &= \frac{\omega_0}{X'_d}\lambda_{dS} - \frac{X_{ad}}{X_{kd}}\frac{\omega_0}{X'_d}\lambda_{dR} \\
i_{dR} &= \frac{X_{ad}}{X_{kd}}\frac{\omega_0}{X'_d}\lambda_{dS} - \frac{X_d}{X'_d}\frac{\omega_0}{X_{kd}}\lambda_{dR}
\end{aligned}
$$

The q- axis is the same.

Torque may be, with these calculations for current, written as:

$$T_e = \frac{3}{2}p\left(\lambda_{dS}i_{qS} - \lambda_{qS}i_{dS}\right) = -\frac{3}{2}p\frac{\omega_0 X_{ad}}{X_{kd}X_d'}\left(\lambda_{dS}\lambda_{qR} - \lambda_{qS}\lambda_{dR}\right)$$

Note that the usual problems with *ordinary* variables hold here: the foregoing expression was written assuming the variables are expressed as *peak* quantities. If RMS is used we must replace 3/2 by 3!

With these, the simulation model is quite straightforward. The state equations are:

$$\frac{d\lambda_{dS}}{dt} = V_{dS} + \omega\lambda_{qS} - R_S i_{dS}$$

$$\frac{d\lambda_{qS}}{dt} = V_{qS} - \omega\lambda_{dS} - R_S i_{qS}$$

$$\frac{d\lambda_{dR}}{dt} = \omega_s\lambda_{qR} - R_R i_{dR}$$

$$\frac{d\lambda_{qR}}{dt} = -\omega_s\lambda_{dR} - R_S i_{qR}$$

$$\frac{d\Omega_m}{dt} = \frac{1}{J}\left(T_e + T_m\right)$$

where the rotor frequency (slip frequency) is:

$$\omega_s = \omega - p\Omega_m$$

For simple simulations and constant excitaion frequency, the choice of coordinate systems is arbitrary, so we can choose something convenient. For example, we might choose to fix the coordinate system to a synchronously rotating frame, so that stator frequency $\omega = \omega_0$. In this case, we could pick the stator voltage to lie on one axis or another. A common choice is $V_d = 0$ and $V_q = V$.

6.8   Control Model

If we are going to turn the machine into a servomotor, we will want to be a bit more sophisticated about our coordinate system. In general, the principle of field-oriented control is much like emulating the function of a DC (commutator) machine. We figure out where the flux is, then inject current to interact most directly with the flux.

As a first step, note that because the two stator flux linkages are the sum of air-gap and leakage flux,

$$\lambda_{dS} = \lambda_{agd} + L_{S\ell}i_{dS}$$

$$\lambda_{qS} = \lambda_{agq} + L_{S\ell}i_{qS}$$

This means that we can re-write torque as:

$$T^e = \frac{3}{2}p\left(\lambda_{agd}i_{qS} - \lambda_{agq}i_{dS}\right)$$

Next, note that the rotor flux is, similarly, related to air-gap flux:

$$\lambda_{agd} = \lambda_{dR} - L_{R\ell} i_{dR}$$
$$\lambda_{agq} = \lambda_{qR} - L_{R\ell} i_{qR}$$

Torque now becomes:

$$T^e = \frac{3}{2} p \left(\lambda_{dR} i_{qS} - \lambda_{qR} i_{dS} \right) - \frac{3}{2} p L_{R\ell} \left(i_{dR} i_{qS} - i_{qR} i_{dS} \right)$$

Now, since the rotor currents could be written as:

$$i_{dR} = \frac{\lambda_{dR}}{L_R} - \frac{M}{L_R} i_{dS}$$
$$i_{qR} = \frac{\lambda_{qR}}{L_R} - \frac{M}{L_R} i_{qS}$$

That second term can be written as:

$$i_{dR} i_{qS} - i_{qR} i_{dS} = \frac{1}{L_R} \left(\lambda_{dR} i_{qS} - \lambda_{qR} i_{dS} \right)$$

So that torque is now:

$$T^e = \frac{3}{2} p \left(1 - \frac{L_{R\ell}}{L_R} \right) \left(\lambda_{dR} i_{qS} - \lambda_{qR} i_{dS} \right) = \frac{3}{2} p \frac{M}{L_R} \left(\lambda_{dR} i_{qS} - \lambda_{qR} i_{dS} \right)$$

6.9 Field-Oriented Strategy:

What is done in field-oriented control is to establish a rotor flux in a known position (usually this position is the d- axis of the transformation) and then put a current on the orthogonal axis (where it will be most effective in producing torque). That is, we will attempt to set

$$\lambda_{dR} = \Lambda_0$$
$$\lambda_{qR} = 0$$

Then torque is produced by applying quadrature-axis current:

$$T^e = \frac{3}{2} p \frac{M}{L_R} \Lambda_0 i_{qS}$$

The process is almost that simple. There are a few details involved in figuring out where the quadrature axis is and how hard to drive the direct axis (magnetizing) current.

Now, suppose we can succeed in putting flux on the right axis, so that $\lambda_{qR} = 0$, then the two rotor voltage equations are:

$$0 = \frac{d\lambda_{dR}}{dt} - \omega_s \lambda_{qR} + r_R I_{dR}$$
$$0 = \frac{d\lambda_{qR}}{dt} + \omega_s \lambda_{dR} + r_R I_{qR}$$

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Now, since the rotor currents are:

$$i_{dR} = \frac{\lambda_{dR}}{L_R} - \frac{M}{L_R}i_{dS}$$
$$i_{qR} = \frac{\lambda_{qR}}{L_R} - \frac{M}{L_R}i_{qS}$$

The voltage expressions become, accounting for the fact that there is no rotor quadrature axis flux:

$$0 = \frac{d\lambda_{dR}}{dt} + r_R\left(\frac{\lambda_{dR}}{L_R} - \frac{M}{L_R}i_{dS}\right)$$
$$0 = \omega_s\lambda_{dR} - r_R\frac{M}{L_R}i_{qS}$$

Noting that the rotor time constant is

$$T_R = \frac{L_R}{r_R}$$

we find:

$$T_R\frac{d\lambda_{dR}}{dt} + \lambda_{dR} = Mi_{dS}$$
$$\omega_s = \frac{M}{T_R}\frac{i_{qS}}{\lambda_{dR}}$$

The first of these two expressions describes the behavior of the direct-axis flux: as one would think, it has a simple first-order relationship with direct-axis stator current. The second expression, which describes slip as a function of quadrature axis current and direct axis flux, actually describes how fast to turn the rotating coordinate system to hold flux on the direct axis.

Now, a real machine application involves phase currents i_a, i_b and i_c, and these must be derived from the model currents i_{dS} and i_{qs}. This is done with, of course, a mathematical operation which uses a transformation angle θ. And that angle is derived from the rotor mechanical speed and computed slip:

$$\theta = \int \left(p\omega_m + \omega_s\right) dt$$

A generally good strategy to make this sort of system work is to measure the three phase currents and derive the direct- and quadrature-axis currents from them. A good estimate of direct-axis flux is made by running direct-axis flux through a first-order filter. The tricky operation involves dividing quadrature axis current by direct axis flux to get slip, but this is now easily done numerically (as are the trigonometric operations required for the rotating coordinate system transformation). An elmentary block diagram of a (possbly) plausible scheme for this is shown in Figure 16.

In this picture we start with commanded values of direct- and quadrature- axis currents, corresponding to flux and torque, respectively. These are translated by a rotating coordinate transformation into commanded phase currents. That transformation (simply the inverse Park's transform) uses the angle q derived as part of the scheme. In some (cheap) implementations of this scheme the commanded currents are used rather than the measured currents to establish the flux and slip.

We have shown the commanded currents i_a^*, etc. as inputs to an "Amplifier". This might be implemented as a PWM current-source, for example, and a tight loop here results in a rather high performance servo system.
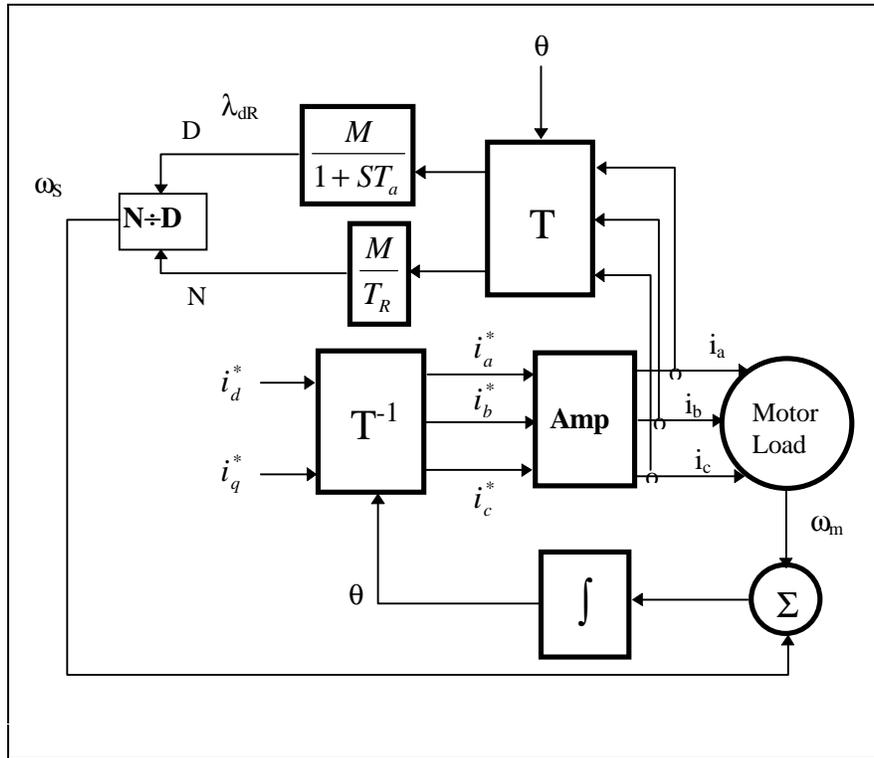
Figure 16: Field Oriented Controller

References

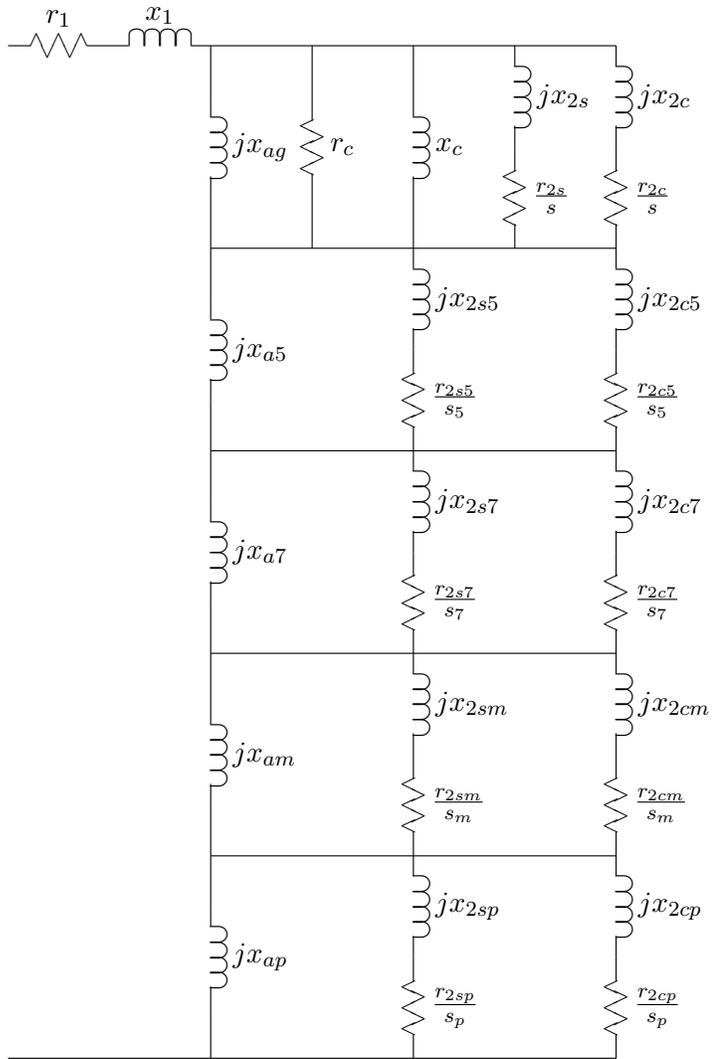[1] P.L. Alger, "Induction Machines", Gordon and Breach, 1969

Figure 17: Extended Equivalent Circuit

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