6.055J / 2.038J The Art of Approximation in Science and Engineering
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$$v \sim \sqrt{\frac{F_{\text{drag}}}{\rho A}}.$$

Since the air density ρ is the same for the large and small cone, the relation simplifies to

$$v \propto \sqrt{\frac{F_{\text{drag}}}{A}}.$$

The cross-sectional areas are easy to measure with a ruler, and the ratio between the small- and large-cone terminal velocities is even easier. The experiment is set up to make the drag force easy to measure: Since the cones fall at their respective terminal velocities, the drag force equals the weight. So

$$v \propto \sqrt{\frac{W}{A}}.$$

Each cone's weight is proportional to its cross-sectional area, because they are geometrically similar and made out of the same piece of paper. With $W \propto A$, the terminal velocity becomes

$$v \propto \sqrt{\frac{A}{A}} = A^0.$$

In other words, the terminal velocity is independent of A, so the small and large cones should fall at the same speed. To test this prediction, I stood on a handy table and dropped the two cones. The fall lasted about two seconds, and they landed within 0.1 s of one another!

5.4.2 Effect of drag on fleas jumping

The drag force

$$F \sim \rho A v^2$$

affects the jumps of small animals more than it affects the jumps of people. A comparison of the energy required for the jump with the energy consumed by drag explains why.

The energy that the animal requires to jump to a height h is mgh, if we use the gravitational potential energy at the top of the jump; or it is $\sim mv^2$, if we use the kinetic energy at takeoff. The energy consumed by drag is

$$E_{\text{drag}} \sim \underbrace{\rho v^2 A}_{F_{\text{drag}}} \times h.$$

The ratio of these energies measures the importance of drag. The ratio is

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho v^2 A h}{m v^2} = \frac{\rho A h}{m}.$$

Since A is the cross-sectional area of the animal, Ah is the volume of air that it sweeps out in the jump, and ρAh is the mass of air swept out in the jump. So the relative importance of drag has a physical interpretation as a ratio of the mass of air displaced to the mass of the animal.

To find how this ratio depends on animal size, rewrite it in terms of the animal's side length l. In terms of side length, $A \sim l^2$ and $m \propto l^3$. What about the jump height h? The simplest analysis predicts that all animals have the same jump height, so $h \propto l^0$. Therefore the numerator ρAh is $\propto l^1$, the denominator m is $\propto l^3$, and

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \propto \frac{l^2}{l^3} = l^{-1}.$$

So, small animals have a large ratio, meaning that drag affects the jumps of small animals more than it affects the jumps of large animals. The missing constant of proportionality means that we cannot say at what size an animal becomes 'small' for the purposes of drag. So the calculation so far cannot tell us whether fleas are included among the small animals.

The jump data, however, substitutes for the missing constant of proportionality. The ratio is

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho Ah}{m} \sim \frac{\rho l^2 h}{\rho_{\text{animal}} l^3}.$$

It simplifies to

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho}{\rho_{\text{animal}}} \frac{h}{l}.$$

As a quick check, verify that the dimensions match. The left side is a ratio of energies, so it is dimensionless. The right side is the product of two dimensionless ratios, so it is also dimensionless. The dimensions match.

Now put in numbers. A density of air is $\rho \sim 1\,\text{kg}\,\text{m}^{-3}$. The density of an animal is roughly the density of water, so $\rho_{\text{animal}} \sim 10^3\,\text{kg}\,\text{m}^{-3}$. The typical jump height – which is where the data substitutes for the constant of proportionality – is 60 cm or roughly 1 m. A flea's length is about 1 mm or $l \sim 10^{-3}$ m. So

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{1\,\text{kg}\,\text{m}^{-3}}{10^3\,\text{kg}\,\text{m}^{-3}} \frac{1\,\text{m}}{10^{-3}\,\text{m}} \sim 1.$$

The ratio being unity means that if a flea would jump to 60 cm, overcoming drag would require roughly as much as energy as would the jump itself in vacuum.

Drag provides a plausible explanation for why fleas do not jump as high as the typical height to which larger animals jump.

5.4.3 Cycling