6.055J / 2.038J The Art of Approximation in Science and Engineering
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Part 2

Symmetry and Invariance

The first part of this book discussed how to organize, and therefore how to manage complexity. The second and third parts discuss how to *eliminate complexity*, with this second part discussing three methods for finding and removing complexity that is not real.

The three methods are proportional reasoning, conservation, and dimensional analysis, and are examples of symmetry reasoning. Symmetry is also a powerful technique on its own, even without using its for the three methods. The next chapter therefore presents general examples of symmetry reasoning, and the following chapters develop this method reasoning into the three methods of proportional reasoning, conservation, and dimensional analysis.

$$A(\omega_{\text{max}}) = \frac{j}{1/Q} = jQ.$$

The factor of Q in the maximum response says that a lightly damped system, where $Q \gg 1$, can reach a high amplitude if you push it at the so-called resonant frequency. The j says that the response at this resonant frequency lags the input by 90 degrees. In other words, the greatest push happens when the velocity, not the displacement, is a maximum.

4.2 Graphical symmetry

The following pictorial problem illustrates symmetry applied to a geometric problem, the traditional domain of symmetry:

> How do you cut an equilateral triangle into two equal halves using the shortest, not-necessarily-straight path?

Here are several candidates among the infinite set of possibilities for the path.

$l = 1/\sqrt{2}$ $l = \sqrt{3}/2$ $l = 1$ $l = $ (a mess)
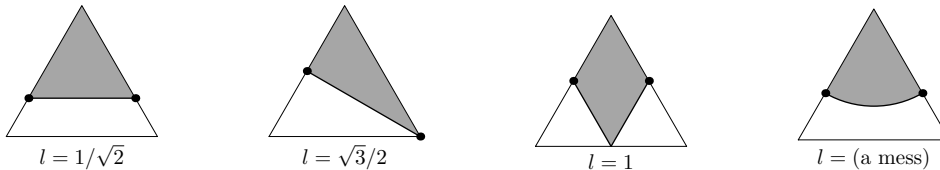
Let's compute the lengths of each bisecting path, with length measured in units of the triangle side. The first candidate encloses an equilateral triangle with one-half the area of the original triangle, so the sides of the smaller, shaded triangle are smaller by a factor of $\sqrt{2}$. Thus the path, being one of those sides, has length $1/\sqrt{2}$. In the second choice, the path is an altitude of the original triangle, which means its length is $\sqrt{3}/2$, so it is longer than the first candidate. The third candidate encloses a diamond made from two small equilateral triangles. Each small triangle has one-fourth the area of the original triangle with side length one, so each small triangle has side length $1/2$. The bisecting path is two sides of a small triangle, so its length is 1. This candidate is longer than the other two.

The fourth candidate is one-sixth of a circle. To find its length, find the radius r of the circle. One-sixth of the circle has one-half the area of the triangle, so

$$\underbrace{\pi r^2}_{A_{\text{circle}}} = 6 \times \frac{1}{2} A_{\text{triangle}} = 6 \times \frac{1}{2} \times \underbrace{\frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2}}_{A_{\text{triangle}}}.$$

Multiplying the pieces gives

$$\pi r^2 = \frac{3\sqrt{3}}{4},$$

and

$$r = \sqrt{\frac{3\sqrt{3}}{4\pi}}.$$

The bisection path is one-sixth of a circle, so its length is

$$l = \frac{2\pi r}{6} = \frac{\pi}{3}\sqrt{\frac{3\sqrt{3}}{4\pi}} = \sqrt{\frac{\pi\sqrt{3}}{12}}.$$

The best previous candidate (the first picture) has length $1/\sqrt{2} = 0.707\ldots$. Does the mess of π and square roots produce a shorter path? Roll the drums... :

$$l = 0.67338\ldots,$$

which is less than $1/\sqrt{2}$. So the circular arc is the best bisection path *so far*. However, is it the best among all possible paths? The arc-length calculation for the circle is messy, and most other paths do not even have a closed form for their arc lengths.

Instead of making elaborate calculations on every path, of which there are many, try **symmetry**, which is the mathematical principle for the three methods in this part of the book. To use symmetry, replicate the triangle six times to make a hexagon, thereby replicating the candidate path as well.
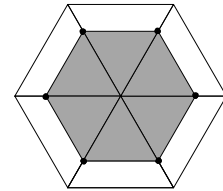
Here is the result of replicating the first candidate where the bisection line goes straight across. The original triangle becomes the large hexagon, and the enclosed half-triangle becomes a smaller hexagon having one-half the area of the large hexagon.

Compare that picture with the result of replicating the circular-arc bisection. The large hexagon is the same as for the last replication, but now the bisected area replicates into a circle. Which path has the shorter perimeter, the shaded hexagon or this circle? The **isoperimetric theorem** says that among all shapes with the same area the circle has the smallest perimeter. Since the circle and the smaller hexagon enclose the same area – which is three times the area of one triangle – the circle has a smaller perimeter than the hexagon, and it has a smaller perimeter than the result of replicating any other bisecting path. So the circular arc is the solution.
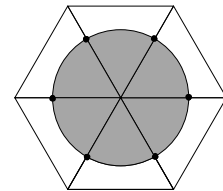
The lesson of this example is that symmetry can remove complexity. The complexity in this problem comes from the edges of the triangle: How much of each edge should be part of the bisected shape? Different paths use different amounts of each edge, and there's no obvious way to deduce the correct amounts. After making the figure symmetric by replicating the triangle into a hexagon, the edges become irrelevant. In the symmetric figure, the question simplifies to finding the shortest path that encloses one-half of the hexagon.

4.3 Heat flow