6.055J / 2.038J The Art of Approximation in Science and Engineering
Spring 2008

6.055J/2.038J (Spring 2008)

Solution set 1

*Do the following warmups and problems. Due in class on **Friday, 15 Feb 2008**.*

***Open universe:** Collaboration, notes, and other sources of information are **encouraged.** However, avoid looking up answers until you solve the problem (or have tried hard). That policy helps you learn the most from the problems.*

***Bring a photocopy to class on the due date**, trade it for a solution set, and figure out or ask me about any confusing points. Your work will be graded lightly: P (made a reasonable effort), D (did not make a reasonable effort), or F (did not turn in).*

Warmups

1. **Air mass**

   Estimate the mass of air in the 6.055J/2.038J classroom and explain your estimate with a tree. If you have not seen the classroom yet, then make more effort to come to lecture (!); meanwhile pictures of the classroom are linked from the course website.

   One way to estimate the mass is to subdivide into the volume of the room and the density of air. The volume of the room subdivides into its length, height, and width. I remember that the density of air is roughly $1\,\mathrm{g}\,\ell^{-1}$ because the value is needed often in estimation problems. Alternatively, you can use a useful fact from chemistry, that one mole of an ideal gas at standard temperature and pressure occupies 22 liters, and combine that fact with the molar mass of air. Using that method, the tree is
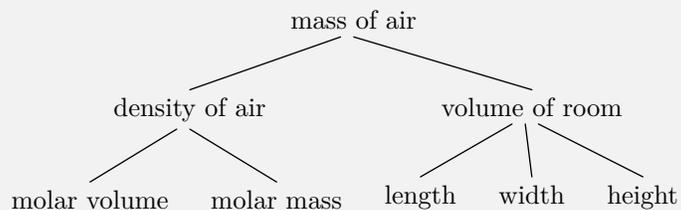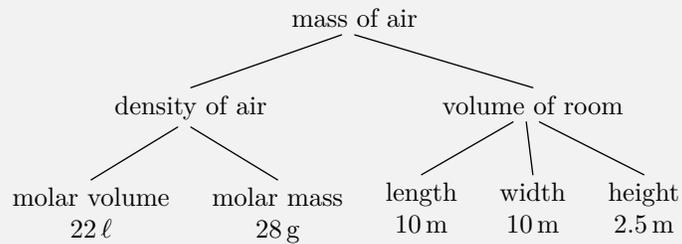

   Now put values at the leaves.

   For the room dimensions, the MIT schedules office webpage gives the room area, but let's estimate the dimensions by eye. Most rooms are 8 or 9 feet high, or about 2.5 m. The room has about 10 rows, spaced around 1 m apart. So the length is about 10 m. The room is roughly square in aspect ratio, so the width is around 10 m as well. [The MIT page says that the area is 768 square feet, or about $77\,\mathrm{m}^2$. Our estimate of $100\,\mathrm{m}^2$ is reasonably accurate.]

   The molar volume for air (like any ideal gas) is 22 liters. The molar mass is, roughly, the molar mass of nitrogen, so about 14 g. But wait, nitrogen is diatomic, so the molar mass is 28 g.

   The tree with values is:

mass of air

density of air   volume of room

molar volume   molar mass   length   width   height
$22\,\ell$ $28\,\mathrm{g}$ $10\,\mathrm{m}$ $10\,\mathrm{m}$ $2.5\,\mathrm{m}$

Now propagate values upward. The volume of the room is $250\,\mathrm{m}^3$. The density of air is roughly $28/22\,\mathrm{g}\,\ell^{-1}$, or roughly $1\,\mathrm{kg}\,\mathrm{m}^{-3}$. So the mass of air in the room is roughly $250\,\mathrm{kg}$ or about 500 pounds.
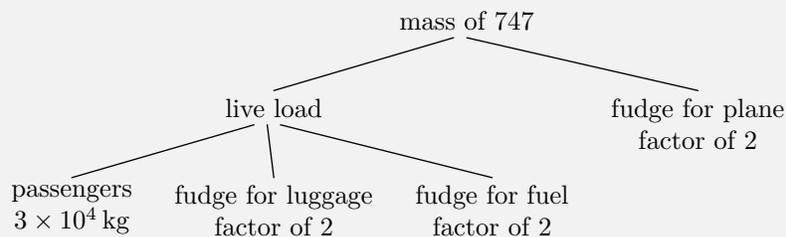
2. 747

Estimate the mass of a full 747 jumbo jet, explaining your estimate using a tree. Then compare with data online. We'll use this value later this semester for estimating the energy costs of flying.

One method – and as usual there are many methods – is to estimate the mass of the passengers and then fudge that value to include the rest of the load (baggage and fuel), and then fudge that value to include the mass of the empty plane. Here is a tree to represent the method:

mass of 747

live load   fudge for plane

passengers   fudge for luggage   fudge for fuel
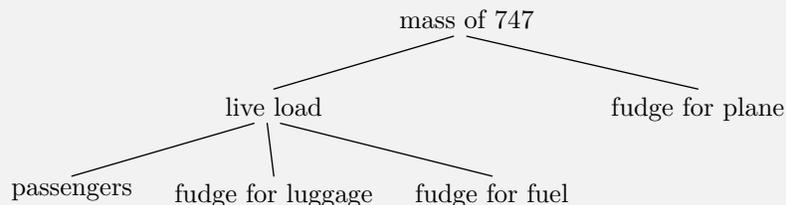
Now let's put values at the leaves. The plane holds about 400 passengers each weighing $70\,\mathrm{kg}$, so $3 \times 10^4\,\mathrm{kg}$. The fuel may be a factor of 2 or 3 larger than the passengers with luggage, although I am not confident about this number. And the plane itself, when empty, might weigh as much as all the load, so include another factor of 2 for the plane itself.

The tree with leaf values is

mass of 747

live load   fudge for plane
                                           factor of 2

passengers   fudge for luggage   fudge for fuel
$3 \times 10^4\,\mathrm{kg}$ factor of 2   factor of 2

Now propagate values upward. The result is

$$m \sim 3 \times 10^4\,\mathrm{kg} \times 2 \times 2 \times 2 \sim 2.5 \times 10^5\,\mathrm{kg}.$$

The actual maximum takeoff weight is (from Wikipedia) $4 \times 10^5\,\mathrm{kg}$. Not too bad (and more accurate than an initial guess without subdividing would have been).

Problems

3. **Random walks and accuracy of divide and conquer**

   Use a coin, a random-number function (in whatever programming language you like), or a table of reasonably random numbers to do the following experiments or their equivalent.

   The first experiment:

   1. Flip a coin 25 times. For each heads move right one step; for each tails, move left one step. At the end of the 25 steps, record your position as a number between −25 and 25.

      > When I flipped a coin 25 times (using a Python program), I ended 3 steps to the right from getting 14 heads and 11 tails.

   2. Repeat the above procedure four times (i.e. three more times), and mark your four ending positions on a number line.

      > In the three repetitions I ended at 7, 3, and −3. The four ending positions are
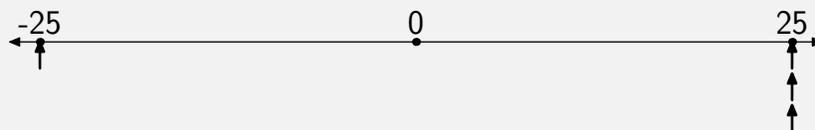   >
      >
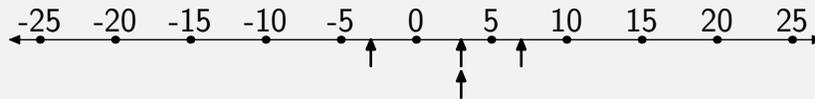
   The second experiment:

   1. Flip a coin once. For heads, move right 25 steps; for tails, move left 25 steps.

      > I ended at +25.

   2. Repeat the above procedure four times (i.e. three more times), and mark your four ending positions on a second number line.

      > On the next repetitions, I ended at +25, −25, and +25. Here are the four ending locations:
      >
      >

   Compare the marks on the two number lines, and explain the relation between this data and the model from lecture for why divide and conquer often reduces errors.

> The marks in the first experiment are clustered much closer to the origin than they are in the second experiment.
>
> The first experiment is analogous to dividing the estimate into 25 equal parts. The errors in the estimate of each part are not likely to point in the same direction. And that's why, when we did the experiment three more times, the results ended up near the origin.
>
> The second experiment is analogous to making the estimate in one gulp, and not subdividing. And that's why the expected error is so large. The moral is to subdivide!

4. **Your turn to create**

   Invent – but do not solve! – an estimation question for which dividing and conquering into assorted subproblems would help solve it (i.e. that would illustrate heterogeneous divide and conquer). To give you an idea for the kinds of problems that might work well, the classic of this genre, due to Fermi, is 'How many piano tuners are there in Chicago?' Other examples are the warmup problems on this problem sheet and the problems that we solved in lecture.

   Particularly interesting or instructive questions will appear on the website or as examples in lecture or the notes (let me know should you *not* want your name attributed in case your question gets selected).