Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2003**

Problem Set 10 Solutions

Solution to Problem 1: Entropy Goes Up

Solution to Problem 1, part a.

The expectation of system energy E_s (the expected value of the energy) is calculated by the following formula.

$$\begin{aligned} E_s &= \sum_i p_{s,i} E_{s,i}(H) \\ &= 0.55\, m_d H - 0.45\, m_d H \\ &= 0.1\, m_d H \end{aligned} \tag{10–1}$$

Solution to Problem 1, part b.

The expectation of the environment energy E_e is found in a similar manner.

$$\begin{aligned} E_e &= \sum_j p_{e,j} E_{e,j}(H) \\ &= 0.25\, m_d H - 0.75\, m_d H \\ &= -0.5\, m_d H \end{aligned} \tag{10–2}$$

Solution to Problem 1, part c.

The environment entropy is calculated via the following formula.

$$\begin{aligned} S_e &= k_B \sum_j p_{e,j} \ln\left(\frac{1}{p_{e,j}}\right) \\ &= k_B \left(0.75 \ln\left(\frac{1}{0.75}\right) + 0.25 \ln\left(\frac{1}{0.25}\right)\right) \\ &= k_B(0.75 \times 0.2877 + 0.25 \times 1.3863) \\ &= k_B(0.2158 + 0.3466) \\ &= 0.5624\, k_B \end{aligned} \tag{10–3}$$

Solution to Problem 1, part d.

The system entropy S_s is calculated in a similar fashion.

$$
\begin{aligned}
S_s &= k_B \sum_i p_{s,i} \ln\left(\frac{1}{p_{s,i}}\right) \\
&= k_B \left(0.45 \ln\left(\frac{1}{0.45}\right) + 0.55 \ln\left(\frac{1}{0.55}\right)\right) \\
&= k_B(0.45 \times 0.7985 + 0.55 \times 0.5978) \\
&= k_B(0.3593 + 0.3288) \\
&= 0.6881\, k_B
\end{aligned} \tag{10–4}
$$

Solution to Problem 1, part e.

No energy leaves the system and environment combined (by definition) so the expectation of the total energy is just the sum of the expectations of the energy of the system and environment.

$$
\begin{aligned}
E_t &= E_s + E_e \\
&= 0.1 m_d H - 0.5 m_d H \\
&= -0.4 m_d H
\end{aligned} \tag{10–5}
$$

Solution to Problem 1, part f.

To find β_t we first use Equation 10.20. Note that $E_{i,j}$=-2, 0,0, or 2 times $m_d H$.

$$
\begin{aligned}
0 &= \sum_{i,j} \left(E_{i,j} - E_t\right) e^{-\beta_t E_{i,j}} \\
&= \sum_{i,j} E_{i,j} e^{-\beta_t E_{i,j}} - E_t \sum_{i,j} e^{-\beta_t E_{i,j}} \\
&= 2 m_d H e^{-2 m_d H \beta_t} - 2 m_d H e^{2 m_d H \beta_t} \\
&\quad + 0.4 m_d H \sum_{i,j} e^{-\beta_t E_{i,j}} \\
&= 2 m_d H e^{-2 m_d H \beta_t} - 2 m_d H e^{2 m_d H \beta_t} \\
&\quad + 0.4 m_d H e^{2 m_d H \beta_t} + 0.8 m_d H + 0.4 m_d H e^{-2 m_d H \beta_t} \\
&= 2.4 m_d H e^{-2 m_d H \beta_t} + 0.8 m_d H - 1.6 m_d H e^{2 m_d H \beta_t} \\
&= 0.8 m_d H \left(3 e^{-2 m_d H \beta_t} + 1 - 2 e^{2 m_d H \beta_t}\right)
\end{aligned} \tag{10–6}
$$

We can rearrange this equation into the following form

$$
\begin{aligned}
0 &= 0.8 m_d H \left(3 e^{-2 m_d H \beta_t} + 1 - 2 e^{2 m_d H \beta_t}\right) \\
0 &= 3 e^{-2 m_d H \beta_t} + 1 - 2 e^{2 m_d H \beta_t} \\
0 &= 3 + e^{2 m_d H \beta_t} - 2 \left(e^{2 m_d H \beta_t}\right)^2 \\
0 &= 2 \left(e^{2 m_d H \beta_t}\right)^2 - e^{2 m_d H \beta_t} - 3
\end{aligned} \tag{10–7}
$$

Letting $x = e^{2 m_d H \beta_t}$ we have

$$
2x^2 - x - 3 = 0 \tag{10–8}
$$

This equation has zeros at $x = -1, 1.5$. Thus $e^{2 m_d H \beta_t} = 1.5$, since it must be greater than zero. Therefore $\beta_t = \ln(1.5)/2 m_d H$.

Solution to Problem 1, part g.

The probabilities are defined as

$$p_{i,j} = \frac{e^{-\beta_t E_{i,j}}}{\sum_{i,j} e^{-\beta_t E_{i,j}}} \tag{10–9}$$

Thus

$$\begin{aligned}
e^{-\beta_t E_{0,0}} &= e^{-(\ln(1.5)/2m_d H)-2m_d H} \\
&= 1.5 && (10–10)\\
e^{-\beta_t E_{0,1}} &= 1 \\
e^{-\beta_t E_{1,0}} &= 1 \\
e^{-\beta_t E_{1,1}} &= e^{-(\ln(1.5)/2m_d H)2m_d H} \\
&= \frac{1}{1.5} && (10–11)
\end{aligned}$$

So

$$\begin{aligned}
p_{0,0} &= \frac{1.5}{1.5 + 0.66 + 2} = 0.36 && (10–12)\\
p_{0,1} &= \frac{1}{1.5 + 0.66 + 2} = 0.24 && (10–13)\\
p_{1,0} &= \frac{1}{1.5 + 0.66 + 2} = 0.24 && (10–14)\\
p_{1,1} &= \frac{0.66}{1.5 + 0.66 + 2} = 0.16 && (10–15)\\
& && (10–16)
\end{aligned}$$

Solution to Problem 1, part h.

The total entropy is

$$\begin{aligned}
S_t &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\
&= k_B \left(0.36 \ln\left(\frac{1}{0.36}\right) + 0.48 \ln\left(\frac{1}{0.24}\right) + 0.16 \ln\left(\frac{1}{0.15}\right)\right) \\
&= k_B\,(0.3677 + 0.6851 + 0.2932) \\
&= 1.6553 k_B && (10–17)
\end{aligned}$$

which is higher than the original entropy, $1.2505 k_B$

Solution to Problem 1, part i.

First let us infer from the four probabilities for the total configuration $p_{t,i,j}$ the probabilities for the two system states $p_{s,i}$.

$$\begin{aligned} p_{s,0} &= p_{t,0,0} + p_{t,0,1} \\ &= 0.36 + 0.24 \\ &= 0.60 \end{aligned} \quad (10\text{–}18)$$

$$\begin{aligned} p_{s,1} &= p_{t,1,0} + p_{t,1,1} \\ &= 0.24 + 0.16 \\ &= 0.40 \end{aligned} \quad (10\text{–}19)$$

Thus the energy is

$$\begin{aligned} E_s &= \sum_i p_{s,i} E_{s,i}(H) \\ &= 0.40\, m_d H - 0.60\, m_d H \\ &= -0.2\, m_d H \end{aligned} \quad (10\text{–}20)$$

Thus we see that exactly half the energy is in the system.

Solution to Problem 1, part j.

The system started out with $0.1 m_d H$ Joules in it, and ended up with $-0.2\, m_d H$ Joules in it. Thus $0.3\, m_d H$ Joules flowed from the environment to the system.

Solution to Problem 2: Energy Conversion (the charge pump)

Solution to Problem 2, part a.

See Figure 10–3.

Solution to Problem 2, part b.

See Figure 10–4.

Solution to Problem 2, part c.

See Figure 10–5.

Solution to Problem 2, part d.

The formula for the charge on a capacitor is as follows.

$$q = \frac{\epsilon_0 A V}{d} \quad (10\text{–}21)$$

You start out with a certain amount of charge on the plate:

$$q_1 = \frac{\epsilon_0 A V_{low}}{d_{min}} \quad (10\text{–}22)$$

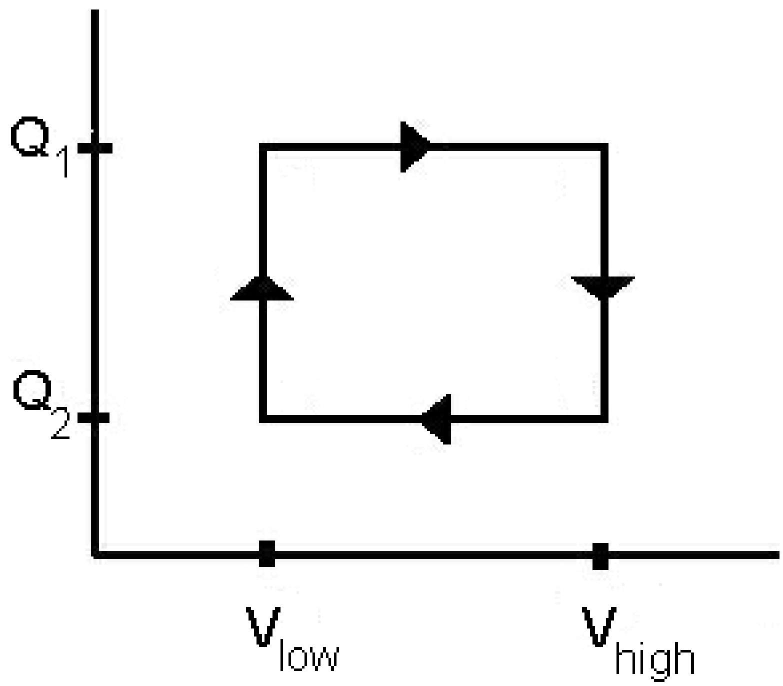

Figure 10–3: Charging Cycle

and finish with a smaller amount of charge:

$$q_2 = \frac{\epsilon_0 A V_{high}}{d_{max}} \tag{10–23}$$

So the charge delivered is:

$$q_0 = q_1 - q_2 = \epsilon_0 A \left(\frac{V_{high}}{d_{max}} - \frac{V_{low}}{d_{min}} \right) \tag{10–24}$$

Solution to Problem 2, part e.

$$E_{high} = V_{high} \epsilon_0 A \left(\frac{V_{high}}{d_{max}} - \frac{V_{low}}{d_{min}} \right) \tag{10–25}$$

Solution to Problem 2, part f.

$$E_{low} = V_{low} \epsilon_0 A \left(\frac{V_{high}}{d_{max}} - \frac{V_{low}}{d_{min}} \right) \tag{10–26}$$

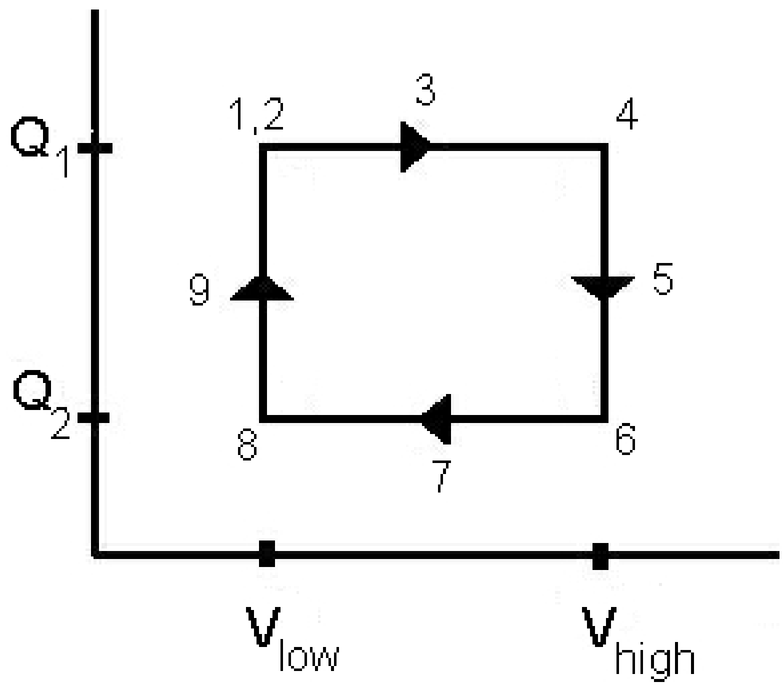

Figure 10–4: Charging Cycle corresponding to user's guide

Solution to Problem 2, part g.

$$
\begin{aligned}
E_{mech} &= E_{high} - E_{low} \\
&= \epsilon_0 A (V_{high} - V_{low}) \left(\frac{V_{high}}{d_{max}} - \frac{V_{low}}{d_{min}} \right)
\end{aligned}
\tag{10–27}
$$

Solution to Problem 2, part h.

Note that a farad is a coulomb per volt.

$$
\begin{aligned}
q_0 &= \epsilon_0 A \left(\frac{V_{low}}{d_{min}} - \frac{V_{high}}{d_{max}} \right) \\
&= (8.854 \times 10^{-12}\mathrm{F/m}) \times (4 \times 10^{-2}\mathrm{m})^2 \times \left(\frac{1.5\mathrm{V}}{0.1 \times 10^{-3}\mathrm{m}} - \frac{9\mathrm{V}}{5 \times 10^{-3}\mathrm{m}} \right) \\
&= 8.854 \times 16 \times 10^{-16} (15000 - 1800)\mathrm{FV} \\
&= 1.869 \times 10^{-10}\mathrm{C}
\end{aligned}
\tag{10–28}
$$

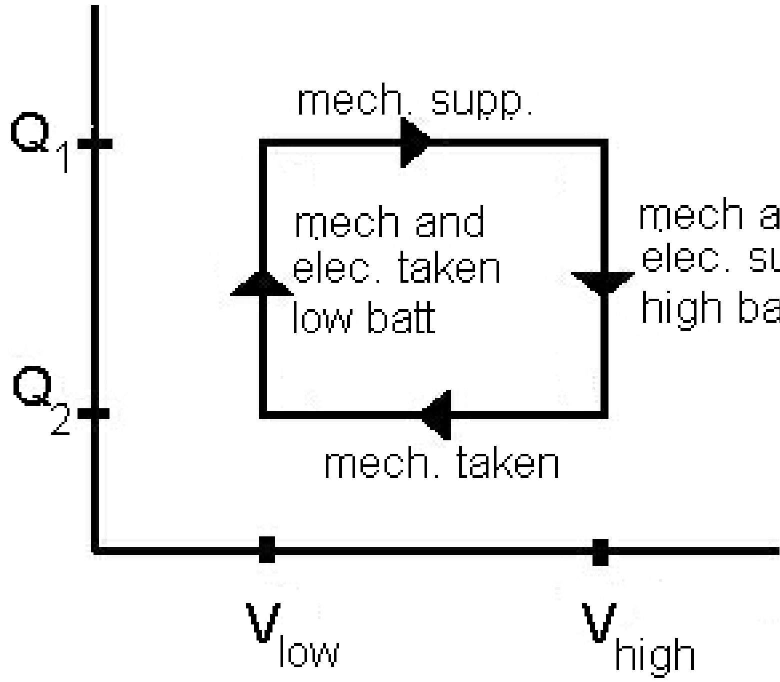

Figure 10–5: Charging Cycle energy movement

The time needed in seconds is numerically equal to the number of charging cycles needed divided by two, which is the desired charge 10^{-9} coulombs divided by the charge per cycle q_0:

$$\frac{10^{-9}}{2 \times 1.869 \times 10^{-10}} = 2.675\text{sec} \tag{10–29}$$

It seems that you can comfortably keep up with the power needed by a nanowatt load.