Problem Set 3

This problem set is due **at 9:00pm** on **Wednesday, February 29, 2012**.

Both exercises and problems should be solved, but *only the problems* should be turned in. Exercises are intended to help you master the course material. Even though you should not turn in the exercise solutions, you are responsible for material covered by the exercises.

Mark the top of the first page of your solution with your name, the course number, the problem number, your recitation section, the date, and the names of any students with whom you collaborated. The homework template (LATEX) is available on the course website.

You will often be called upon to "give an algorithm" to solve a certain problem. Your write-up should take the form of a short essay. A topic paragraph should summarize the problem you are solving and what your results are. The body of the essay should provide the following:

1. A description of the algorithm in English and, if helpful, pseudo-code.

2. At least one worked example or diagram to show more precisely how your algorithm works.

3. A proof (or indication) of the correctness of the algorithm.

4. An analysis of the running time of the algorithm.

Remember, your goal is to communicate. Full credit will be given only to correct solutions *that are described clearly*. Convoluted and opaque descriptions will receive lower marks.

Exercise 3-1. Do Exercise 30.1-2 on page 905 of CLRS.

Exercise 3-2. Do Exercise 30.1-3 on page 905 of CLRS.

Exercise 3-3. Do Exercise 30.1-7 on page 906 of CLRS.

Exercise 3-4. Do Exercise 30.2-7 on page 914 of CLRS.

Problem 3-1. Electric Potential Problem

According to Coulomb's Law, the electric potential created by a point charge q, at a distance r from the charge, is:

$$V_E = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

There are n charges in a square uniform grid of $m \times m$ points. For $i = 1, 2, \ldots, n$, the charge i has a charge value q_i and is located at grid point (x_i, y_i), where x_i and y_i are integers $0 \le x_i, y_i < m$. For each grid point (x, y) not occupied by a charge, the ***effective electric potential*** is:

$$V(x, y) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} \ .$$

The electric potential problem is to find the effective electric potential at each of the $m^2 - n$ grid points unoccupied by a charge.

(a) Describe a simple $O(m^2 n)$ time algorithm to solve the problem.

(b) Let $\mathbb{Z}_{2m-1} = \{0, 1, \ldots, 2m - 2\}$. Find two functions $f, g : \mathbb{Z}_{2m-1} \times \mathbb{Z}_{2m-1} \to \mathbb{R}$ such that the potential at (x, y) for $0 \le x, y < m$ equals the convolution of f and g:

$$\begin{aligned} V(x, y) &= (f \otimes g)(x, y) \\ &= \sum_{x'=0}^{2m-2} \sum_{y'=0}^{2m-2} f(x', y') \cdot g(x - x', y - y') \ . \end{aligned}$$

Importantly, in this definition $x - x'$ and $y - y'$ are computed in the additive group $\mathbb{Z}_{2m-1}$, which is a fancy way of saying they are computed modulo $2m - 1$.

For positive integer k, the ***discrete Fourier transform*** of a function $h : \mathbb{Z}_k \times \mathbb{Z}_k \to \mathbb{R}$ is the function $\widehat{h} : \mathbb{Z}_k \times \mathbb{Z}_k \to \mathbb{C}$ defined as follows:

$$\widehat{h}(a, b) = \frac{1}{k^2} \sum_{x=0}^{k-1} \sum_{y=0}^{k-1} h(x, y) \omega_k^{-ax - by} \ ,$$

where ω_k is a kth root of unity.

The corresponding ***inverse discrete Fourier transform*** of $\widehat{h} : \mathbb{Z}_k \times \mathbb{Z}_k \to \mathbb{C}$ is defined as follows:

$$h(x, y) = \sum_{a=0}^{k-1} \sum_{b=0}^{k-1} \widehat{h}(a, b) \omega_k^{ax + by} \ .$$

(c) Prove that for any two functions $f, g : \mathbb{Z}_k \times \mathbb{Z}_k \to \mathbb{R}$ and for any point $(a, b) \in \mathbb{Z}_k \times \mathbb{Z}_k$, we have

$$\widehat{(f \otimes g)}(a, b) = k^2 \cdot \widehat{f}(a, b) \cdot \widehat{g}(a, b) \ .$$

(d) Design an $O(k^2 \lg k)$ time algorithm to compute the discrete Fourier transform and its inverse.

(e) Design an $O(m^2 \lg m)$ time algorithm to solve the electric potential problem for a grid of size $m \times m$.

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