

Chapter 16

Counting

16.1 Why Count?

Are there two different subsets of the ninety 25-digit numbers shown below that have the same sum—for example, maybe the sum of the numbers in the first column is equal to the sum of the numbers in the second column?

0020480135385502964448038	3171004832173501394113017
5763257331083479647409398	8247331000042995311646021
0489445991866915676240992	3208234421597368647019265
5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113
6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365
6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149
6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246
6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815
6814428944266874963488274	8772321203608477245851154
1470029452721203587686214	4080505804577801451363100
6870852945543886849147881	8791422161722582546341091
1578271047286257499433886	4167283461025702348124920
6914955508120950093732397	9062628024592126283973285
1638243921852176243192354	423599683112377788211249
6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220
7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427
7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190
7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530
7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910
7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856
7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348
7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372
7712154432211912882310511	9602413424619187112552264
3075514410490975920315348	5384358126771794128356947

7858918664240262356610010	9631217114906129219461111
8149436716871371161932035	3157693105325111284321993
3111474985252793452860017	5439211712248901995423441
7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458
8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044
5692168374637019617423712	8176063831682536571306791

Finding two subsets with the same sum may seem like an silly puzzle, but solving problems like this turns out to be useful, for example in finding good ways to fit packages into shipping containers and in decoding secret messages.

The answer to the question turns out to be “yes.” Of course this would be easy to confirm just by showing two subsets with the same sum, but that turns out to be kind of hard to do. So before we put a lot of effort into finding such a pair, it would be nice to be sure there were some. Fortunately, it *is* very easy to see why there is such a pair—or at least it will be easy once we have developed a few simple rules for counting things.

The Contest to Find Two Sets with the Same Sum

One term, Eric Lehman, a 6.042 instructor who contributed to many parts of this book, offered a \$100 prize for being the first 6.042 student to actually *find* two different subsets of the above ninety 25-digit numbers that have the same sum. Eric didn’t expect to have to pay off this bet, but he underestimated the ingenuity and initiative of 6.042 students.

One computer science major wrote a program that cleverly searched only among a reasonably small set of “plausible” sets, sorted them by their sums, and actually found a couple with the same sum. He won the prize. A few days later, a math major figured out how to reformulate the sum problem as a “lattice basis reduction” problem; then he found a software package implementing an efficient basis reduction procedure, and using it, he very quickly found lots of pairs of subsets with the same sum. He didn’t win the prize, but he got a standing ovation from the class—staff included.

Counting seems easy enough: 1, 2, 3, 4, etc. This direct approach works well for counting simple things—like your toes—and may be the only approach for extremely complicated things with no identifiable structure. However, subtler methods can help you count many things in the vast middle ground, such as:

- The number of different ways to select a dozen doughnuts when there are five varieties available.
- The number of 16-bit numbers with exactly 4 ones.

Counting is useful in computer science for several reasons:

- Determining the time and storage required to solve a computational problem—a central objective in computer science—often comes down to solving a counting problem.
- Counting is the basis of probability theory, which plays a central role in all sciences, including computer science.
- Two remarkable proof techniques, the “pigeonhole principle” and “combinatorial proof,” rely on counting. These lead to a variety of interesting and useful insights.

We’re going to present a lot of rules for counting. These rules are actually theorems, but most of them are pretty obvious anyway, so we’re not going to focus on proving them. Our objective is to teach you simple counting as a practical skill, like integration.

16.2 Counting One Thing by Counting Another

How do you count the number of people in a crowded room? You could count heads, since for each person there is exactly one head. Alternatively, you could count ears and divide by two. Of course, you might have to adjust the calculation if someone lost an ear in a pirate raid or someone was born with three ears. The point here is that you can often *count one thing by counting another*, though some fudge factors may be required.

In more formal terms, every counting problem comes down to determining the size of some set. The *size* or *cardinality* of a finite set, S , is the number of elements in it and is denoted $|S|$. In these terms, we’re claiming that we can often find the size of one set by finding the size of a related set. We’ve already seen a general statement of this idea in the Mapping Rule of Lemma [4.8.2](#).

16.2.1 The Bijection Rule

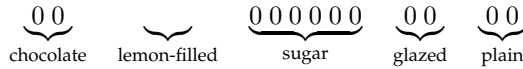
We’ve already implicitly used the Bijection Rule of Lemma [3](#) a lot. For example, when we studied Stable Marriage and Bipartite Matching, we assumed the obvious fact that if we can pair up all the girls at a dance with all the boys, then there must be an equal number of each. If we needed to be explicit about using the Bijection Rule, we could say that A was the set of boys, B was the set of girls, and the bijection between them was how they were paired.

The Bijection Rule acts as a magnifier of counting ability; if you figure out the size of one set, then you can immediately determine the sizes of many other sets via bijections. For example, let’s return to two sets mentioned earlier:

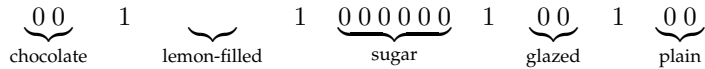
A = all ways to select a dozen doughnuts when five varieties are available

B = all 16-bit sequences with exactly 4 ones

Let's consider a particular element of set A :



We've depicted each doughnut with a 0 and left a gap between the different varieties. Thus, the selection above contains two chocolate doughnuts, no lemon-filled, six sugar, two glazed, and two plain. Now let's put a 1 into each of the four gaps:

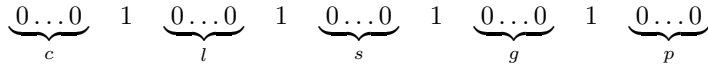


We've just formed a 16-bit number with exactly 4 ones— an element of B !

This example suggests a bijection from set A to set B : map a dozen doughnuts consisting of:

c chocolate, l lemon-filled, s sugar, g glazed, and p plain

to the sequence:



The resulting sequence always has 16 bits and exactly 4 ones, and thus is an element of B . Moreover, the mapping is a bijection; every such bit sequence is mapped to by exactly one order of a dozen doughnuts. Therefore, $|A| = |B|$ by the Bijection Rule!

This demonstrates the magnifying power of the bijection rule. We managed to prove that two very different sets are actually the same size— even though we don't know exactly how big either one is. But as soon as we figure out the size of one set, we'll immediately know the size of the other.

This particular bijection might seem frighteningly ingenious if you've not seen it before. But you'll use essentially this same argument over and over, and soon you'll consider it routine.

16.2.2 Counting Sequences

The Bijection Rule lets us count one thing by counting another. This suggests a general strategy: get really good at counting just a *few* things and then use bijections to count *everything else*. This is the strategy we'll follow. In particular, we'll get really good at counting *sequences*. When we want to determine the size of some other set T , we'll find a bijection from T to a set of sequences S . Then we'll use our super-ninja sequence-counting skills to determine $|S|$, which immediately gives us $|T|$. We'll need to hone this idea somewhat as we go along, but that's pretty much the plan!

16.2.3 The Sum Rule

Linus allocates his big sister Lucy a quota of 20 crabby days, 40 irritable days, and 60 generally surly days. On how many days can Lucy be out-of-sorts one way or another? Let set C be her crabby days, I be her irritable days, and S be her generally surly. In these terms, the answer to the question is $|C \cup I \cup S|$. Now assuming that she is permitted at most one bad quality each day, the size of this union of sets is given by the *Sum Rule*:

Rule 1 (Sum Rule). *If A_1, A_2, \dots, A_n are disjoint sets, then:*

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$$

Thus, according to Linus' budget, Lucy can be out-of-sorts for:

$$\begin{aligned} |C \cup I \cup S| &= |C| + |I| + |S| \\ &= 20 + 40 + 60 \\ &= 120 \text{ days} \end{aligned}$$

Notice that the Sum Rule holds only for a union of *disjoint* sets. Finding the size of a union of intersecting sets is a more complicated problem that we'll take up later.

16.2.4 The Product Rule

The *Product Rule* gives the size of a product of sets. Recall that if P_1, P_2, \dots, P_n are sets, then

$$P_1 \times P_2 \times \dots \times P_n$$

is the set of all sequences whose first term is drawn from P_1 , second term is drawn from P_2 and so forth.

Rule 2 (Product Rule). *If P_1, P_2, \dots, P_n are sets, then:*

$$|P_1 \times P_2 \times \dots \times P_n| = |P_1| \cdot |P_2| \cdots |P_n|$$

Unlike the sum rule, the product rule does not require the sets P_1, \dots, P_n to be disjoint. For example, suppose a *daily diet* consists of a breakfast selected from set B , a lunch from set L , and a dinner from set D :

$$\begin{aligned} B &= \{\text{pancakes, bacon and eggs, bagel, Doritos}\} \\ L &= \{\text{burger and fries, garden salad, Doritos}\} \\ D &= \{\text{macaroni, pizza, frozen burrito, pasta, Doritos}\} \end{aligned}$$

Then $B \times L \times D$ is the set of all possible daily diets. Here are some sample elements:

$$\begin{aligned} &(\text{pancakes, burger and fries, pizza}) \\ &(\text{bacon and eggs, garden salad, pasta}) \\ &(\text{Doritos, Doritos, frozen burrito}) \end{aligned}$$

The Product Rule tells us how many different daily diets are possible:

$$\begin{aligned} |B \times L \times D| &= |B| \cdot |L| \cdot |D| \\ &= 4 \cdot 3 \cdot 5 \\ &= 60 \end{aligned}$$

16.2.5 Putting Rules Together

Few counting problems can be solved with a single rule. More often, a solution is a flurry of sums, products, bijections, and other methods. Let's look at some examples that bring more than one rule into play.

Counting Passwords

The sum and product rules together are useful for solving problems involving passwords, telephone numbers, and license plates. For example, on a certain computer system, a valid password is a sequence of between six and eight symbols. The first symbol must be a letter (which can be lowercase or uppercase), and the remaining symbols must be either letters or digits. How many different passwords are possible?

Let's define two sets, corresponding to valid symbols in the first and subsequent positions in the password.

$$\begin{aligned} F &= \{a, b, \dots, z, A, B, \dots, Z\} \\ S &= \{a, b, \dots, z, A, B, \dots, Z, 0, 1, \dots, 9\} \end{aligned}$$

In these terms, the set of all possible passwords is:

$$(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)$$

Thus, the length-six passwords are in set $F \times S^5$, the length-seven passwords are in $F \times S^6$, and the length-eight passwords are in $F \times S^7$. Since these sets are disjoint, we can apply the Sum Rule and count the total number of possible passwords as follows:

$$\begin{aligned} |(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)| &= |F \times S^5| + |F \times S^6| + |F \times S^7| && \text{Sum Rule} \\ &= |F| \cdot |S|^5 + |F| \cdot |S|^6 + |F| \cdot |S|^7 && \text{Product Rule} \\ &= 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7 \\ &\approx 1.8 \cdot 10^{14} \text{ different passwords} \end{aligned}$$

Subsets of an n -element Set

How many different subsets of an n -element set X are there? For example, the set $X = \{x_1, x_2, x_3\}$ has eight different subsets:

$$\begin{array}{cccc} \emptyset & \{x_1\} & \{x_2\} & \{x_1, x_2\} \\ \{x_3\} & \{x_1, x_3\} & \{x_2, x_3\} & \{x_1, x_2, x_3\} \end{array}$$

There is a natural bijection from subsets of X to n -bit sequences. Let x_1, x_2, \dots, x_n be the elements of X . Then a particular subset of X maps to the sequence (b_1, \dots, b_n) where $b_i = 1$ if and only if x_i is in that subset. For example, if $n = 10$, then the subset $\{x_2, x_3, x_5, x_7, x_{10}\}$ maps to a 10-bit sequence as follows:

$$\begin{array}{r} \text{subset: } \{ \quad x_2, \quad x_3, \quad x_5, \quad x_7, \quad x_{10} \quad \} \\ \text{sequence: } (\quad 0, \quad 1, \quad 1, \quad 0, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0, \quad 1 \quad) \end{array}$$

We just used a bijection to transform the original problem into a question about sequences —*exactly according to plan!* Now if we answer the sequence question, then we've solved our original problem as well.

But how many different n -bit sequences are there? For example, there are 8 different 3-bit sequences:

$$\begin{array}{cccc} (0, 0, 0) & (0, 0, 1) & (0, 1, 0) & (0, 1, 1) \\ (1, 0, 0) & (1, 0, 1) & (1, 1, 0) & (1, 1, 1) \end{array}$$

Well, we can write the set of all n -bit sequences as a product of sets:

$$\underbrace{\{0, 1\} \times \{0, 1\} \times \dots \times \{0, 1\}}_{n \text{ terms}} = \{0, 1\}^n$$

Then Product Rule gives the answer:

$$\begin{aligned} |\{0, 1\}^n| &= |\{0, 1\}|^n \\ &= 2^n \end{aligned}$$

This means that the number of subsets of an n -element set X is also 2^n . We'll put this answer to use shortly.

16.2.6 Problems

Practice Problems

Problem 16.1.

How many ways are there to select k out of n books on a shelf so that there are always at least 3 unselected books between selected books? (Assume n is large enough for this to be possible.)

Class Problems

Problem 16.2.

A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)

- 2 characters – letters or numbers (big shot plate)

Let L be the set of all possible license plates.

- (a) Express L in terms of

$$\begin{aligned}\mathcal{A} &= \{A, B, C, \dots, Z\} \\ \mathcal{D} &= \{0, 1, 2, \dots, 9\}\end{aligned}$$

using unions (\cup) and set products (\times).

- (b) Compute $|L|$, the number of different license plates, using the sum and product rules.

Problem 16.3. (a) How many of the billion numbers in the range from 1 to 10^9 contain the digit 1? (*Hint: How many don't?*)

(b) There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15-bit strings with exactly 6 ones.

Problem 16.4.

- (a) Let $\mathcal{S}_{n,k}$ be the possible nonnegative integer solutions to the inequality

$$x_1 + x_2 + \dots + x_k \leq n. \quad (16.1)$$

That is

$$\mathcal{S}_{n,k} ::= \{(x_1, x_2, \dots, x_k) \in \mathbb{N}^k \mid (16.1) \text{ is true}\}.$$

Describe a bijection between $\mathcal{S}_{n,k}$ and the set of binary strings with n zeroes and k ones.

- (b) Let $\mathcal{L}_{n,k}$ be the length k weakly increasing sequences of nonnegative integers $\leq n$. That is

$$\mathcal{L}_{n,k} ::= \{(y_1, y_2, \dots, y_k) \in \mathbb{N}^k \mid y_1 \leq y_2 \leq \dots \leq y_k \leq n\}.$$

Describe a bijection between $\mathcal{L}_{n,k}$ and $\mathcal{S}_{n,k}$.

Problem 16.5.

An n -vertex *numbered tree* is a tree whose vertex set is $\{1, 2, \dots, n\}$ for some $n > 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers from 1 to n obtained by the following recursive process:

If there are more than two vertices left, write down the *father* of the largest leaf^a, delete this *leaf*, and continue this process on the resulting smaller tree.

If there are only two vertices left, then stop —the code is complete.

^aThe necessarily unique node adjacent to a leaf is called its *father*.

For example, the codes of a couple of numbered trees are shown in the Figure 16.2.

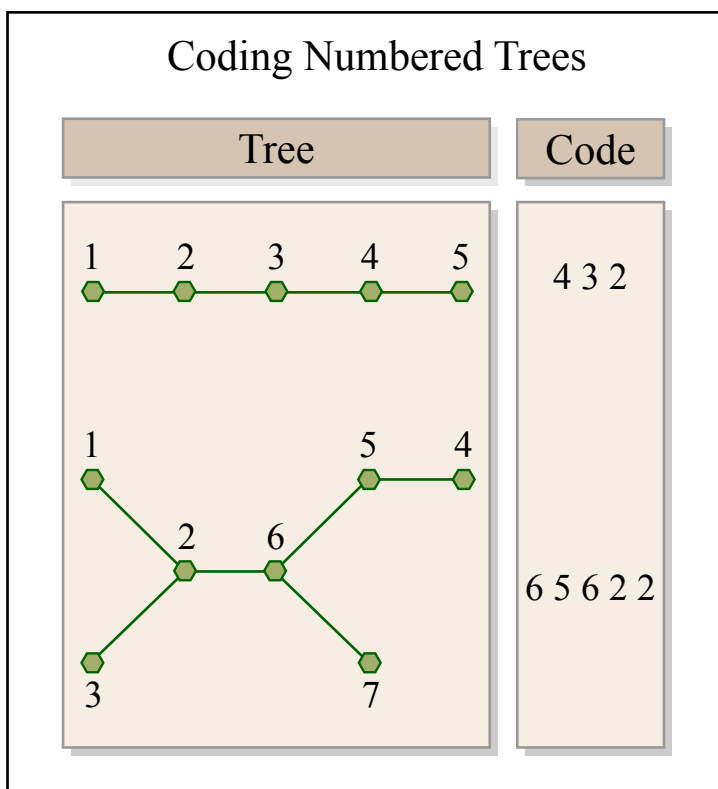


Image by MIT OpenCourseWare.

Figure 16.1:

(a) Describe a procedure for reconstructing a numbered tree from its code.

(b) Conclude there is a bijection between the n -vertex numbered trees and $\{1, \dots, n\}^{n-2}$, and state how many n -vertex numbered trees there are.

Homework Problems

Problem 16.6.

Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o, u appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

(b) How many ways are there to order the 26 letters of the alphabet so that there are *at least two* consonants immediately following each vowel?

(c) In how many different ways can $2n$ students be paired up?

(d) Two n -digit sequences of digits $0, 1, \dots, 9$ are said to be of the *same type* if the digits of one are a permutation of the digits of the other. For $n = 8$, for example, the sequences 03088929 and 00238899 are the same type. How many types of n -digit integers are there?

Problem 16.7.

In a standard 52-card deck, each card has one of thirteen *ranks* in the set, R , and one of four *suits* in the set, S , where

$$R ::= \{A, 2, \dots, 10, J, Q, K\},$$

$$S ::= \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

A 5-card *hand* is a set of five distinct cards from the deck.

For each part describe a bijection between a set that can easily be counted using the Product and Sum Rules of Ch. 16.2, and the set of hands matching the specification. *Give bijections, not numerical answers.*

For instance, consider the set of 5-card hands containing all 4 suits. Each such hand must have 2 cards of one suit. We can describe a bijection between such hands and the set $S \times R_2 \times R^3$ where R_2 is the set of two-element subsets of R . Namely, an element

$$(s, \{r_1, r_2\}, (r_3, r_4, r_5)) \in S \times R_2 \times R^3$$

indicates

1. the repeated suit, $s \in S$,
2. the set, $\{r_1, r_2\} \in R_2$, of ranks of the cards of suit, s , and
3. the ranks (r_3, r_4, r_5) of remaining three cards, listed in increasing suit order where $\clubsuit < \diamondsuit < \heartsuit < \spadesuit$.

For example,

$$(\clubsuit, \{10, A\}, (J, J, 2)) \longleftrightarrow \{A\clubsuit, 10\clubsuit, J\diamondsuit, J\heartsuit, 2\spadesuit\}.$$

- (a) A single pair of the same rank (no 3-of-a-kind, 4-of-a-kind, or second pair).
- (b) Three or more aces.

16.3 The Pigeonhole Principle

Here is an old puzzle:

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

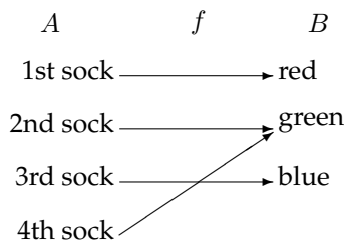
For example, picking out three socks is not enough; you might end up with one red, one green, and one blue. The solution relies on the *Pigeonhole Principle*, which is a friendly name for the contrapositive of the injective case 2 of the Mapping Rule of Lemma 4.8.2. Let's write it down:

If $|X| > |Y|$, then no total function¹ $f : X \rightarrow Y$ is injective.

And now rewrite it again to eliminate the word “injective.”

Rule 3 (Pigeonhole Principle). *If $|X| > |Y|$, then for every total function $f : X \rightarrow Y$, there exist two different elements of X that are mapped to the same element of Y .*

What this abstract mathematical statement has to do with selecting footwear under poor lighting conditions is maybe not obvious. However, let A be the set of socks you pick out, let B be the set of colors available, and let f map each sock to its color. The Pigeonhole Principle says that if $|A| > |B| = 3$, then at least two elements of A (that is, at least two socks) must be mapped to the same element of B (that is, the same color). For example, one possible mapping of four socks to three colors is shown below.



¹This Mapping Rule actually applies even if f is a total injective *relation*.

Therefore, four socks are enough to ensure a matched pair.

Not surprisingly, the pigeonhole principle is often described in terms of pigeons:

If there are more pigeons than holes they occupy, then at least two pigeons must be in the same hole.

In this case, the pigeons form set A , the pigeonholes are set B , and f describes which hole each pigeon flies into.

Mathematicians have come up with many ingenious applications for the pigeonhole principle. If there were a cookbook procedure for generating such arguments, we'd give it to you. Unfortunately, there isn't one. One helpful tip, though: when you try to solve a problem with the pigeonhole principle, the key is to clearly identify three things:

1. The set A (the pigeons).
2. The set B (the pigeonholes).
3. The function f (the rule for assigning pigeons to pigeonholes).

16.3.1 Hairs on Heads

There are a number of generalizations of the pigeonhole principle. For example:

Rule 4 (Generalized Pigeonhole Principle). *If $|X| > k \cdot |Y|$, then every total function $f : X \rightarrow Y$ maps at least $k + 1$ different elements of X to the same element of Y .*

For example, if you pick two people at random, surely they are extremely unlikely to have *exactly* the same number of hairs on their heads. However, in the remarkable city of Boston, Massachusetts there are actually *three* people who have exactly the same number of hairs! Of course, there are many bald people in Boston, and they all have zero hairs. But we're talking about non-bald people; say a person is non-bald if they have at least ten thousand hairs on their head.

Boston has about 500,000 non-bald people, and the number of hairs on a person's head is at most 200,000. Let A be the set of non-bald people in Boston, let $B = \{10,000, 10,001, \dots, 200,000\}$, and let f map a person to the number of hairs on his or her head. Since $|A| > 2|B|$, the Generalized Pigeonhole Principle implies that at least three people have exactly the same number of hairs. We don't know who they are, but we know they exist!

16.3.2 Subsets with the Same Sum

We asserted that two different subsets of the ninety 25-digit numbers listed on the first page have the same sum. This actually follows from the Pigeonhole Principle. Let A be the collection of all subsets of the 90 numbers in the list. Now the sum of any subset of numbers is at most $90 \cdot 10^{25}$, since there are only 90 numbers and every

25-digit number is less than 10^{25} . So let B be the set of integers $\{0, 1, \dots, 90 \cdot 10^{25}\}$, and let f map each subset of numbers (in A) to its sum (in B).

We proved that an n -element set has 2^n different subsets. Therefore:

$$\begin{aligned} |A| &= 2^{90} \\ &\geq 1.237 \times 10^{27} \end{aligned}$$

On the other hand:

$$\begin{aligned} |B| &= 90 \cdot 10^{25} + 1 \\ &\leq 0.901 \times 10^{27} \end{aligned}$$

Both quantities are enormous, but $|A|$ is a bit greater than $|B|$. This means that f maps at least two elements of A to the same element of B . In other words, by the Pigeonhole Principle, two different subsets must have the same sum!

Notice that this proof gives no indication *which* two sets of numbers have the same sum. This frustrating variety of argument is called a *nonconstructive proof*.

Sets with Distinct Subset Sums

How can we construct a set of n positive integers such that all its subsets have *distinct* sums? One way is to use powers of two:

$$\{1, 2, 4, 8, 16\}$$

This approach is so natural that one suspects all other such sets must involve larger numbers. (For example, we could safely replace 16 by 17, but not by 15.) Remarkably, there are examples involving *smaller* numbers. Here is one:

$$\{6, 9, 11, 12, 13\}$$

One of the top mathematicians of the Twentieth Century, Paul Erdős, conjectured in 1931 that there are no such sets involving *significantly* smaller numbers. More precisely, he conjectured that the largest number must be $> c2^n$ for some constant $c > 0$. He offered \$500 to anyone who could prove or disprove his conjecture, but the problem remains unsolved.

16.3.3 Problems

Class Problems

Problem 16.8.

Solve the following problems using the pigeonhole principle. For each problem,

try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) Every MIT ID number starts with a 9 (we think). Suppose that each of the 75 students in 6.042 sums the nine digits of his or her ID number. Explain why two people must arrive at the same sum.

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

(d) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \dots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k .

Homework Problems

Problem 16.9.

Pigeon Huntin'

(a) Show that any odd integer x in the range $10^9 < x < 2 \cdot 10^9$ containing all ten digits $0, 1, \dots, 9$ must have consecutive even digits. *Hint:* What can you conclude about the parities of the first and last digit?

(b) Show that there are 2 vertices of equal degree in any finite undirected graph with $n \geq 2$ vertices. *Hint:* Cases conditioned upon the existence of a degree zero vertex.

Problem 16.10.

Show that for any set of 201 positive integers less than 300, there must be two whose quotient is a power of three (with no remainder).

16.4 The Generalized Product Rule

We realize everyone has been working pretty hard this term, and we're considering awarding some prizes for *truly exceptional* coursework. Here are some possible categories:

Best Administrative Critique We asserted that the quiz was closed-book. On the cover page, one strong candidate for this award wrote, "There is no book."

Awkward Question Award "Okay, the left sock, right sock, and pants are in an antichain, but how— even with assistance— could I put on all three at once?"

Best Collaboration Statement Inspired by a student who wrote "I worked alone" on Quiz 1.

In how many ways can, say, three different prizes be awarded to n people? This is easy to answer using our strategy of translating the problem about awards into a problem about sequences. Let P be the set of n people in 6.042. Then there is a bijection from ways of awarding the three prizes to the set $P^3 ::= P \times P \times P$. In particular, the assignment:

“person x wins prize #1, y wins prize #2, and z wins prize #3”

maps to the sequence (x, y, z) . By the Product Rule, we have $|P^3| = |P|^3 = n^3$, so there are n^3 ways to award the prizes to a class of n people.

But what if the three prizes must be awarded to *different* students? As before, we could map the assignment

“person x wins prize #1, y wins prize #2, and z wins prize #3”

to the triple $(x, y, z) \in P^3$. But this function is *no longer a bijection*. For example, no valid assignment maps to the triple (Dave, Dave, Becky) because Dave is not allowed to receive two awards. However, there *is* a bijection from prize assignments to the set:

$$S = \{(x, y, z) \in P^3 \mid x, y, \text{ and } z \text{ are different people}\}$$

This reduces the original problem to a problem of counting sequences. Unfortunately, the Product Rule is of no help in counting sequences of this type because the entries depend on one another; in particular, they must all be different. However, a slightly sharper tool does the trick.

Rule 5 (Generalized Product Rule). *Let S be a set of length- k sequences. If there are:*

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- n_3 possible third entries for each combination of first and second entries, etc.

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

In the awards example, S consists of sequences (x, y, z) . There are n ways to choose x , the recipient of prize #1. For each of these, there are $n - 1$ ways to choose y , the recipient of prize #2, since everyone except for person x is eligible. For each combination of x and y , there are $n - 2$ ways to choose z , the recipient of prize #3, because everyone except for x and y is eligible. Thus, according to the Generalized Product Rule, there are

$$|S| = n \cdot (n - 1) \cdot (n - 2)$$

ways to award the 3 prizes to different people.

16.4.1 Defective Dollars

A dollar is *defective* if some digit appears more than once in the 8-digit serial number. If you check your wallet, you'll be sad to discover that defective dollars are all-too-common. In fact, how common are *nondefective* dollars? Assuming that the digit portions of serial numbers all occur equally often, we could answer this question by computing:

$$\text{fraction dollars that are nondefective} = \frac{\# \text{ of serial #'s with all digits different}}{\text{total \# of serial #'s}}$$

Let's first consider the denominator. Here there are no restrictions; there are 10 possible first digits, 10 possible second digits, 10 third digits, and so on. Thus, the total number of 8-digit serial numbers is 10^8 by the Product Rule.

Next, let's turn to the numerator. Now we're not permitted to use any digit twice. So there are still 10 possible first digits, but only 9 possible second digits, 8 possible third digits, and so forth. Thus, by the Generalized Product Rule, there are

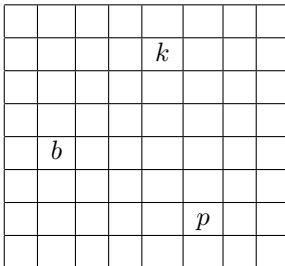
$$\begin{aligned} 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 &= \frac{10!}{2} \\ &= 1,814,400 \end{aligned}$$

serial numbers with all digits different. Plugging these results into the equation above, we find:

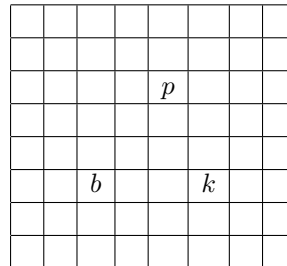
$$\begin{aligned} \text{fraction dollars that are nondefective} &= \frac{1,814,400}{100,000,000} \\ &= 1.8144\% \end{aligned}$$

16.4.2 A Chess Problem

In how many different ways can we place a pawn (p), a knight (k), and a bishop (b) on a chessboard so that no two pieces share a row or a column? A valid configuration is shown below on the left, and an invalid configuration is shown on the right.



valid



invalid

First, we map this problem about chess pieces to a question about sequences. There is a bijection from configurations to sequences

$$(r_p, c_p, r_k, c_k, r_b, c_b)$$

where r_p , r_k , and r_b are distinct rows and c_p , c_k , and c_b are distinct columns. In particular, r_p is the pawn's row, c_p is the pawn's column, r_k is the knight's row, etc. Now we can count the number of such sequences using the Generalized Product Rule:

- r_p is one of 8 rows
- c_p is one of 8 columns
- r_k is one of 7 rows (any one but r_p)
- c_k is one of 7 columns (any one but c_p)
- r_b is one of 6 rows (any one but r_p or r_k)
- c_b is one of 6 columns (any one but c_p or c_k)

Thus, the total number of configurations is $(8 \cdot 7 \cdot 6)^2$.

16.4.3 Permutations

A *permutation* of a set S is a sequence that contains every element of S exactly once. For example, here are all the permutations of the set $\{a, b, c\}$:

$$\begin{array}{lll} (a, b, c) & (a, c, b) & (b, a, c) \\ (b, c, a) & (c, a, b) & (c, b, a) \end{array}$$

How many permutations of an n -element set are there? Well, there are n choices for the first element. For each of these, there are $n - 1$ remaining choices for the second element. For every combination of the first two elements, there are $n - 2$ ways to choose the third element, and so forth. Thus, there are a total of

$$n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$$

permutations of an n -element set. In particular, this formula says that there are $3! = 6$ permutations of the 3-element set $\{a, b, c\}$, which is the number we found above.

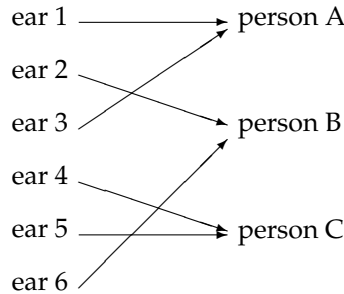
Permutations will come up again in this course approximately 1.6 bazillion times. In fact, permutations are the reason why factorial comes up so often and why we taught you Stirling's approximation:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

16.5 The Division Rule

Counting ears and dividing by two is a silly way to count the number of people in a room, but this approach is representative of a powerful counting principle.

A *k-to-1 function* maps exactly k elements of the domain to every element of the codomain. For example, the function mapping each ear to its owner is 2-to-1:



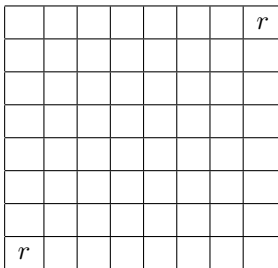
Similarly, the function mapping each finger to its owner is 10-to-1, and the function mapping each finger and toe to its owner is 20-to-1. The general rule is:

Rule 6 (Division Rule). *If $f : A \rightarrow B$ is k -to-1, then $|A| = k \cdot |B|$.*

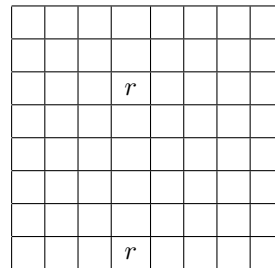
For example, suppose A is the set of ears in the room and B is the set of people. There is a 2-to-1 mapping from ears to people, so by the Division Rule $|A| = 2 \cdot |B|$ or, equivalently, $|B| = |A|/2$, expressing what we knew all along: the number of people is half the number of ears. Unlikely as it may seem, many counting problems are made much easier by initially counting every item multiple times and then correcting the answer using the Division Rule. Let's look at some examples.

16.5.1 Another Chess Problem

In how many different ways can you place two identical rooks on a chessboard so that they do not share a row or column? A valid configuration is shown below on the left, and an invalid configuration is shown on the right.



valid



invalid

Let A be the set of all sequences

$$(r_1, c_1, r_2, c_2)$$

where r_1 and r_2 are distinct rows and c_1 and c_2 are distinct columns. Let B be the set of all valid rook configurations. There is a natural function f from set A to set B ; in particular, f maps the sequence (r_1, c_1, r_2, c_2) to a configuration with one rook in row r_1 , column c_1 and the other rook in row r_2 , column c_2 .

But now there's a snag. Consider the sequences:

$$(1, 1, 8, 8) \quad \text{and} \quad (8, 8, 1, 1)$$

The first sequence maps to a configuration with a rook in the lower-left corner and a rook in the upper-right corner. The second sequence maps to a configuration with a rook in the upper-right corner and a rook in the lower-left corner. The problem is that those are two different ways of describing the *same* configuration! In fact, this arrangement is shown on the left side in the diagram above.

More generally, the function f maps exactly two sequences to *every* board configuration; that is f is a 2-to-1 function. Thus, by the quotient rule, $|A| = 2 \cdot |B|$. Rearranging terms gives:

$$\begin{aligned} |B| &= \frac{|A|}{2} \\ &= \frac{(8 \cdot 7)^2}{2} \end{aligned}$$

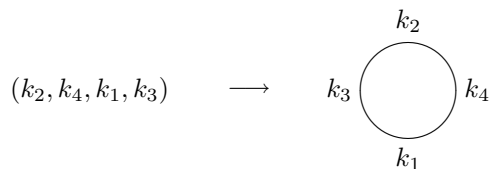
On the second line, we've computed the size of A using the General Product Rule just as in the earlier chess problem.

16.5.2 Knights of the Round Table

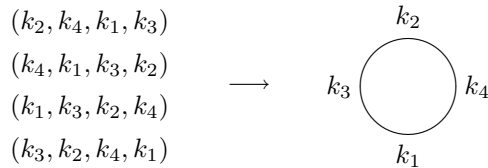
In how many ways can King Arthur seat n different knights at his round table? Two seatings are considered equivalent if one can be obtained from the other by rotation. For example, the following two arrangements are equivalent:



Let A be all the permutations of the knights, and let B be the set of all possible seating arrangements at the round table. We can map each permutation in set A to a circular seating arrangement in set B by seating the first knight in the permutation anywhere, putting the second knight to his left, the third knight to the left of the second, and so forth all the way around the table. For example:



This mapping is actually an n -to-1 function from A to B , since all n cyclic shifts of the original sequence map to the same seating arrangement. In the example, $n = 4$ different sequences map to the same seating arrangement:



Therefore, by the division rule, the number of circular seating arrangements is:

$$\begin{aligned}
 |B| &= \frac{|A|}{n} \\
 &= \frac{n!}{n} \\
 &= (n-1)!
 \end{aligned}$$

Note that $|A| = n!$ since there are $n!$ permutations of n knights.

16.5.3 Problems

Class Problems

Problem 16.11.

Your 6.006 tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your TA has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k -to-1 mapping for what k ?

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j -to-1 for what j ?

- (c) How many group assignments are possible?
- (d) In how many ways can $3n$ students be broken up into n groups of 3?

Problem 16.12.

A pizza house is having a promotional sale. Their commercial reads:

We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's 22, 369, 621 different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to 22, 369, 621. Unfortunately, $(2^9)^3/3!$ is obviously not an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Problem 16.13.

Answer the following questions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

(b) An r -permutation of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{lll} (a, b) & (a, c) & (a, d) \\ (b, a) & (b, c) & (b, d) \\ (c, a) & (c, b) & (c, d) \\ (d, a) & (d, b) & (d, c) \end{array}$$

How many r -permutations of an n -element set are there? Express your answer using factorial notation.

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \dots, p\}$, where $p \geq n^2$?

Exam Problems**Problem 16.14.**

Suppose that two identical 52-card decks are mixed together. Write a simple formula for the number of 104-card double-deck mixes that are possible.

16.6 Counting Subsets

How many k -element subsets of an n -element set are there? This question arises all the time in various guises:

- In how many ways can I select 5 books from my collection of 100 to bring on vacation?
- How many different 13-card Bridge hands can be dealt from a 52-card deck?
- In how many ways can I select 5 toppings for my pizza if there are 14 available toppings?

This number comes up so often that there is a special notation for it:

$$\binom{n}{k} ::= \text{the number of } k\text{-element subsets of an } n\text{-element set.}$$

The expression $\binom{n}{k}$ is read “ n choose k .” Now we can immediately express the answers to all three questions above:

- I can select 5 books from 100 in $\binom{100}{5}$ ways.
- There are $\binom{52}{13}$ different Bridge hands.
- There are $\binom{14}{5}$ different 5-topping pizzas, if 14 toppings are available.

16.6.1 The Subset Rule

We can derive a simple formula for the n -choose- k number using the Division Rule. We do this by mapping any permutation of an n -element set $\{a_1, \dots, a_n\}$ into a k -element subset simply by taking the first k elements of the permutation. That is, the permutation $a_1 a_2 \dots a_n$ will map to the set $\{a_1, a_2, \dots, a_k\}$.

Notice that any other permutation with the same first k elements a_1, \dots, a_k in any order and the same remaining elements $n - k$ elements in any order will also map to this set. What’s more, a permutation can only map to $\{a_1, a_2, \dots, a_k\}$ if its first k elements are the elements a_1, \dots, a_k in some order. Since there are $k!$ possible permutations of the first k elements and $(n - k)!$ permutations of the remaining elements, we conclude from the Product Rule that exactly $k!(n - k)!$ permutations of the n -element set map to the particular subset, S . In other words, the mapping from permutations to k -element subsets is $k!(n - k)!$ -to-1.

But we know there are $n!$ permutations of an n -element set, so by the Division Rule, we conclude that

$$n! = k!(n - k)! \binom{n}{k}$$

which proves:

Rule 7 (Subset Rule). *The number,*

$$\binom{n}{k},$$

of k -element subsets of an n -element set is

$$\frac{n!}{k!(n-k)!}.$$

Notice that this works even for 0-element subsets: $n!/0!n! = 1$. Here we use the fact that $0!$ is a *product* of 0 terms, which by convention equals 1. (A *sum* of zero terms equals 0.)

16.6.2 Bit Sequences

How many n -bit sequences contain exactly k ones? We've already seen the straightforward bijection between subsets of an n -element set and n -bit sequences. For example, here is a 3-element subset of $\{x_1, x_2, \dots, x_8\}$ and the associated 8-bit sequence:

$$\left\{ \begin{array}{cccccccc} x_1, & & & x_4, & x_5 & & & \\ (1, & 0, & 0, & 1, & 1, & 0, & 0, & 0) \end{array} \right\}$$

Notice that this sequence has exactly 3 ones, each corresponding to an element of the 3-element subset. More generally, the n -bit sequences corresponding to a k -element subset will have exactly k ones. So by the Bijection Rule,

The number of n -bit sequences with exactly k ones is $\binom{n}{k}$.

16.7 Sequences with Repetitions

16.7.1 Sequences of Subsets

Choosing a k -element subset of an n -element set is the same as splitting the set into a pair of subsets: the first subset of size k and the second subset consisting of the remaining $n - k$ elements. So the Subset Rule can be understood as a rule for counting the number of such splits into pairs of subsets.

We can generalize this to splits into more than two subsets. Namely, let A be an n -element set and k_1, k_2, \dots, k_m be nonnegative integers whose sum is n . A (k_1, k_2, \dots, k_m) -split of A is a sequence

$$(A_1, A_2, \dots, A_m)$$

where the A_i are pairwise disjoint² subsets of A and $|A_i| = k_i$ for $i = 1, \dots, m$.

²That is $A_i \cap A_j = \emptyset$ whenever $i \neq j$. Another way to say this is that no element appears in more than one of the A_i 's.

The same reasoning used to explain the Subset Rule extends directly to a rule for counting the number of splits into subsets of given sizes.

Rule 8 (Subset Split Rule). *The number of (k_1, k_2, \dots, k_m) -splits of an n -element set is*

$$\binom{n}{k_1, \dots, k_m} ::= \frac{n!}{k_1! k_2! \cdots k_m!}$$

The proof of this Rule is essentially the same as for the Subset Rule. Namely, we map any permutation $a_1 a_2 \dots a_n$ of an n -element set, A , into a (k_1, k_2, \dots, k_m) -split by letting the 1st subset in the split be the first k_1 elements of the permutation, the 2nd subset of the split be the next k_2 elements, \dots , and the m th subset of the split be the final k_m elements of the permutation. This map is a $k_1! k_2! \cdots k_m!$ -to-1 from the $n!$ permutations to the (k_1, k_2, \dots, k_m) -splits of A , and the Subset Split Rule now follows from the Division Rule.

16.7.2 The Bookkeeper Rule

We can also generalize our count of n -bit sequences with k -ones to counting length n sequences of letters over an alphabet with more than two letters. For example, how many sequences can be formed by permuting the letters in the 10-letter word BOOKKEEPER?

Notice that there are 1 B, 2 O's, 2 K's, 3 E's, 1 P, and 1 R in BOOKKEEPER. This leads to a straightforward bijection between permutations of BOOKKEEPER and $(1, 2, 2, 3, 1, 1)$ -splits of $\{1, \dots, n\}$. Namely, map a permutation to the sequence of sets of positions where each of the different letters occur.

For example, in the permutation BOOKKEEPER itself, the B is in the 1st position, the O's occur in the 2nd and 3rd positions, K's in 4th and 5th, the E's in the 6th, 7th and 9th, P in the 8th, and R is in the 10th position, so BOOKKEEPER maps to

$$(\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7, 9\}, \{8\}, \{10\}).$$

From this bijection and the Subset Split Rule, we conclude that the number of ways to rearrange the letters in the word BOOKKEEPER is:

$$\frac{\overbrace{10!}^{\text{total letters}}}{\underbrace{1!}_{\text{B's}} \underbrace{2!}_{\text{O's}} \underbrace{2!}_{\text{K's}} \underbrace{3!}_{\text{E's}} \underbrace{1!}_{\text{P's}} \underbrace{1!}_{\text{R's}}}$$

This example generalizes directly to an exceptionally useful counting principle which we will call the

Rule 9 (Bookkeeper Rule). *Let l_1, \dots, l_m be distinct elements. The number of sequences with k_1 occurrences of l_1 , and k_2 occurrences of l_2 , \dots , and k_m occurrences of l_m is*

$$\frac{(k_1 + k_2 + \dots + k_m)!}{k_1! k_2! \dots k_m!}$$

Example. 20-Mile Walks.

I'm planning a 20-mile walk, which should include 5 northward miles, 5 eastward miles, 5 southward miles, and 5 westward miles. How many different walks are possible?

There is a bijection between such walks and sequences with 5 N's, 5 E's, 5 S's, and 5 W's. By the Bookkeeper Rule, the number of such sequences is:

$$\frac{20!}{5!^4}$$

16.7.3 A Word about Words

Someday you might refer to the Subset Split Rule or the Bookkeeper Rule in front of a roomful of colleagues and discover that they're all staring back at you blankly. This is not because they're dumb, but rather because we made up the name "Bookkeeper Rule". However, the rule is excellent and the name is apt, so we suggest that you play through: "You know? The Bookkeeper Rule? Don't you guys know *anything???*"

The Bookkeeper Rule is sometimes called the "formula for permutations with indistinguishable objects." The size k subsets of an n -element set are sometimes called k -combinations. Other similar-sounding descriptions are "combinations with repetition, permutations with repetition, r -permutations, permutations with indistinguishable objects," and so on. However, the counting rules we've taught you are sufficient to solve all these sorts of problems without knowing this jargon, so we won't burden you with it.

16.7.4 Problems

Class Problems

Problem 16.15.

The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word *BOOKKEEPER*.

- (a) In how many ways can you arrange the letters in the word *POKE*?
- (b) In how many ways can you arrange the letters in the word BO_1O_2K ? Observe that we have subscripted the O's to make them distinct symbols.
- (c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in *BOOK* by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	\dots
BO_2O_1K	
\dots	

- (d) What kind of mapping is this, young grasshopper?
- (e) In light of the Division Rule, how many arrangements are there of $BOOK$?
- (f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?
- (g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.
- (h) What kind of mapping is this?
- (i) So how many arrangements are there of the letters in $KEEPER$?
- (j) *Now you are ready to face the BOOKKEEPER!*
- How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?
- (k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?
- (l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?
- (m) How many arrangements of $BOOKKEEPER$ are there?

*Remember well what you have learned: subscripts on, subscripts off.
This is the Tao of Bookkeeper.*

- (n) How many arrangements of $VOODOODOLL$ are there?
- (o) How many length 52 sequences of digits contain exactly 17 two's, 23 fives, and 12 nines?

16.8 Magic Trick

There is a Magician and an Assistant. The Assistant goes into the audience with a deck of 52 cards while the Magician looks away.³

³ There are 52 cards in a standard deck. Each card has a *suit* and a *rank*. There are four suits:

♠ (spades) ♥ (hearts) ♣ (clubs) ♦ (diamonds)

Five audience members each select one card from the deck. The Assistant then gathers up the five cards and holds up four of them so the Magician can see them. The Magician concentrates for a short time and then correctly names the secret, fifth card!

Since we don't really believe the Magician can read minds, we know the Assistant has somehow communicated the secret card to the Magician. Since real Magicians and Assistants are not to be trusted, we can expect that the Assistant would illegitimately signal the Magician with coded phrases or body language, but they don't have to cheat in this way. In fact, the Magician and Assistant could be kept out of sight of each other while some audience member holds up the 4 cards designated by the Assistant for the Magician to see.

Of course, without cheating, there is still an obvious way the Assistant can communicate to the Magician: he can choose any of the $4! = 24$ permutations of the 4 cards as the order in which to hold up the cards. However, this alone won't quite work: there are 48 cards remaining in the deck, so the Assistant doesn't have enough choices of orders to indicate exactly what the secret card is (though he could narrow it down to two cards).

16.8.1 The Secret

The method the Assistant can use to communicate the fifth card exactly is a nice application of what we know about counting and matching.

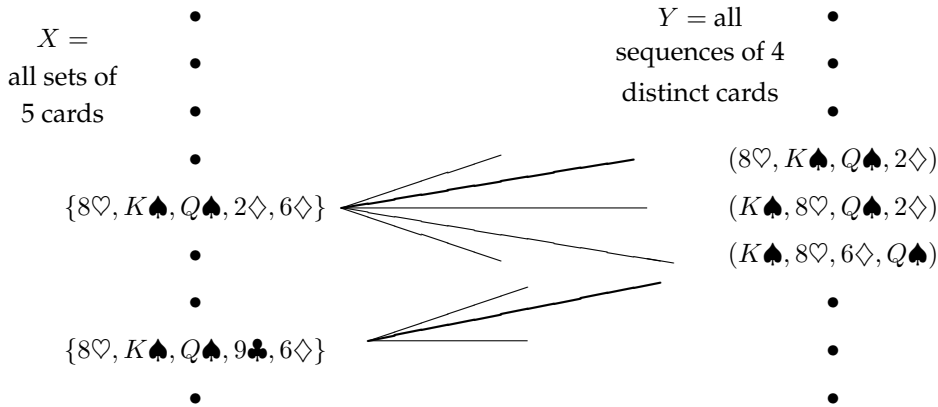
The Assistant really has another legitimate way to communicate: he can choose *which of the five cards to keep hidden*. Of course, it's not clear how the Magician could determine which of these five possibilities the Assistant selected by looking at the four visible cards, but there is a way, as we'll now explain.

The problem facing the Magician and Assistant is actually a bipartite matching problem. Put all the *sets* of 5 cards in a collection, X , on the left. And put all the sequences of 4 distinct cards in a collection, Y , on the right. These are the two sets of vertices in the bipartite graph. There is an edge between a set of 5 cards and a sequence of 4 if every card in the sequence is also in the set. In other words, if the audience selects a set of cards, then the Assistant must reveal a sequence of cards that is adjacent in the bipartite graph. Some edges are shown in the diagram below.

And there are 13 ranks, listed here from lowest to highest:

Ace Jack Queen King
 $A, 2, 3, 4, 5, 6, 7, 8, 9, J, Q, K$

Thus, for example, $8\heartsuit$ is the 8 of hearts and $A\spadesuit$ is the ace of spades.



For example,

$$\{8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit, 6\diamondsuit\} \tag{16.2}$$

is an element of X on the left. If the audience selects this set of 5 cards, then there are many different 4-card sequences on the right in set Y that the Assistant could choose to reveal, including $(8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit)$, $(K\spadesuit, 8\heartsuit, Q\spadesuit, 2\diamondsuit)$, and $(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit)$.

What the Magician and his Assistant need to perform the trick is a *matching* for the X vertices. If they agree in advance on some matching, then when the audience selects a set of 5 cards, the Assistant reveals the matching sequence of 4 cards. The Magician uses the reverse of the matching to find the audience’s chosen set of 5 cards, and so he can name the one not already revealed.

For example, suppose the Assistant and Magician agree on a matching containing the two bold edges in the diagram above. If the audience selects the set

$$\{8\heartsuit, K\spadesuit, Q\spadesuit, 9\clubsuit, 6\diamondsuit\}, \tag{16.3}$$

then the Assistant reveals the corresponding sequence

$$(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit). \tag{16.4}$$

Using the matching, the Magician sees that the hand (16.3) is matched to the sequence (16.4), so he can name the one card in the corresponding set not already revealed, namely, the $9\clubsuit$. Notice that the fact that the sets are *matched*, that is, that different sets are paired with *distinct* sequences, is essential. For example, if the audience picked the previous hand (16.2), it would be possible for the Assistant to reveal the same sequence (16.4), but he better not do that: if he did, then the Magician would have no way to tell if the remaining card was the $9\clubsuit$ or the $2\diamondsuit$.

So how can we be sure the needed matching can be found? The reason is that each vertex on the left has degree $5 \cdot 4! = 120$, since there are five ways to select the card kept secret and there are $4!$ permutations of the remaining 4 cards. In addition, each vertex on the right has degree 48, since there are 48 possibilities for the fifth card. So this graph is *degree-constrained* according to Definition 10.6.5, and therefore satisfies Hall’s matching condition.

In fact, this reasoning shows that the Magician could still pull off the trick if 120 cards were left instead of 48, that is, the trick would work with a deck as large as 124 different cards —without any magic!

16.8.2 The Real Secret

But wait a minute! It's all very well in principle to have the Magician and his Assistant agree on a matching, but how are they supposed to remember a matching with $\binom{52}{5} = 2,598,960$ edges? For the trick to work in practice, there has to be a way to match hands and card sequences mentally and on the fly.

We'll describe one approach. As a running example, suppose that the audience selects:

10♥ 9♦ 3♥ Q♠ J♦

- The Assistant picks out two cards of the same suit. In the example, the assistant might choose the 3♥ and 10♥.
- The Assistant locates the ranks of these two cards on the cycle shown below:

	K	A	2	
	Q			3
	J			4
	10			5
	9			6
		8	7	

For any two distinct ranks on this cycle, one is always between 1 and 6 hops clockwise from the other. For example, the 3♥ is 6 hops clockwise from the 10♥.

- The more counterclockwise of these two cards is revealed first, and the other becomes the secret card. Thus, in our example, the 10♥ would be revealed, and the 3♥ would be the secret card. Therefore:
 - The suit of the secret card is the same as the suit of the first card revealed.
 - The rank of the secret card is between 1 and 6 hops clockwise from the rank of the first card revealed.
- All that remains is to communicate a number between 1 and 6. The Magician and Assistant agree beforehand on an ordering of all the cards in the deck from smallest to largest such as:

A♣ A♦ A♥ A♠ 2♣ 2♦ 2♥ 2♠ ... K♥ K♠

The order in which the last three cards are revealed communicates the number according to the following scheme:

$$\begin{aligned} (\text{small, medium, large}) &= 1 \\ (\text{small, large, medium}) &= 2 \\ (\text{medium, small, large}) &= 3 \\ (\text{medium, large, small}) &= 4 \\ (\text{large, small, medium}) &= 5 \\ (\text{large, medium, small}) &= 6 \end{aligned}$$

In the example, the Assistant wants to send 6 and so reveals the remaining three cards in large, medium, small order. Here is the complete sequence that the Magician sees:

$$10\heartsuit \quad Q\spadesuit \quad J\diamondsuit \quad 9\diamondsuit$$

- The Magician starts with the first card, $10\heartsuit$, and hops 6 ranks clockwise to reach $3\heartsuit$, which is the secret card!

So that's how the trick can work with a standard deck of 52 cards. On the other hand, Hall's Theorem implies that the Magician and Assistant can *in principle* perform the trick with a deck of up to 124 cards. It turns out that there is a method which they could actually learn to use with a reasonable amount of practice for a 124 card deck (see *The Best Card Trick* by Michael Kleber).

16.8.3 Same Trick with Four Cards?

Suppose that the audience selects only *four* cards and the Assistant reveals a sequence of *three* to the Magician. Can the Magician determine the fourth card?

Let X be all the sets of four cards that the audience might select, and let Y be all the sequences of three cards that the Assistant might reveal. Now, on one hand, we have

$$|X| = \binom{52}{4} = 270,725$$

by the Subset Rule. On the other hand, we have

$$|Y| = 52 \cdot 51 \cdot 50 = 132,600$$

by the Generalized Product Rule. Thus, by the Pigeonhole Principle, the Assistant must reveal the *same* sequence of three cards for at least

$$\left\lceil \frac{270,725}{132,600} \right\rceil = 3$$

different four-card hands. This is bad news for the Magician: if he sees that sequence of three, then there are at least three possibilities for the fourth card which he cannot distinguish. So there is no legitimate way for the Assistant to communicate exactly what the fourth card is!

16.8.4 Problems

Class Problems

Problem 16.16. (a) Show that the Magician could not pull off the trick with a deck larger than 124 cards.

Hint: Compare the number of 5-card hands in an n -card deck with the number of 4-card sequences.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of 124 cards.

Hint: Hall's Theorem and degree-constrained (10.6.5) graphs.

Problem 16.17.

The Magician can determine the 5th card in a poker hand when his Assisant reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your Assisant reveals the other 7 cards.

Homework Problems

Problem 16.18.

Section 16.8.3 explained why it is not possible to perform a four-card variant of the hidden-card magic trick with one card hidden. But the Magician and her Assistant are determined to find a way to make a trick like this work. They decide to change the rules slightly: instead of the Assistant lining up the three unhidden cards for the Magician to see, he will line up all four cards with one card face down and the other three visible. We'll call this the *face-down four-card trick*.

For example, suppose the audience members had selected the cards $9\heartsuit$, $10\diamondsuit$, $A\clubsuit$, $5\clubsuit$. Then the Assistant could choose to arrange the 4 cards in any order so long as one is face down and the others are visible. Two possibilities are:

$A\clubsuit$?	$10\diamondsuit$	$5\clubsuit$
?	$5\clubsuit$	$9\heartsuit$	$10\diamondsuit$

(a) Explain why there must be a bipartite matching which will in theory allow the Magician and Assistant to perform the face-down four-card trick.

(b) There is actually a simple way to perform the face-down four-card trick.

Case 1. *there are two cards with the same suit:* Say there are two ♠ cards. The Assistant proceeds as in the original card trick: he puts one of the ♠ cards *face up as the first card*. He will place the second ♠ card *face down*. He then uses a permutation of the face down card and the remaining two face up cards to code the offset of the face down card from the first card.

Case 2. *all four cards have different suits:* Assign numbers 0, 1, 2, 3 to the four suits in some agreed upon way. The Assistant computes, s , the sum modulo 4 of the ranks of the four cards, and chooses the card with suit s to be placed *face down as the first card*. He then uses a permutation of the remaining three face-up cards to code the rank of the face down card.^a

^aThis elegant method was devised in Fall '09 by student Katie E Everett.

Explain how in Case 2. the Magician can determine the face down card from the cards the Assistant shows her.

(c) Explain how any method for performing the face-down four-card trick can be adapted to perform the regular (5-card hand, show 4 cards) with a 52-card deck consisting of the usual 52 cards along with a 53rd card call the *joker*.

16.9 Counting Practice: Poker Hands

Five-Card Draw is a card game in which each player is initially dealt a *hand*, a subset of 5 cards. (Then the game gets complicated, but let's not worry about that.) The number of different hands in Five-Card Draw is the number of 5-element subsets of a 52-element set, which is 52 choose 5:

$$\text{total \# of hands} = \binom{52}{5} = 2,598,960$$

Let's get some counting practice by working out the number of hands with various special properties.

16.9.1 Hands with a Four-of-a-Kind

A *Four-of-a-Kind* is a set of four cards with the same rank. How many different hands contain a Four-of-a-Kind? Here are a couple examples:

$$\begin{aligned} & \{ 8\spadesuit, 8\diamondsuit, Q\heartsuit, 8\heartsuit, 8\clubsuit \} \\ & \{ A\clubsuit, 2\clubsuit, 2\heartsuit, 2\diamondsuit, 2\spadesuit \} \end{aligned}$$

As usual, the first step is to map this question to a sequence-counting problem. A hand with a Four-of-a-Kind is completely described by a sequence specifying:

1. The rank of the four cards.
2. The rank of the extra card.
3. The suit of the extra card.

Thus, there is a bijection between hands with a Four-of-a-Kind and sequences consisting of two distinct ranks followed by a suit. For example, the three hands above are associated with the following sequences:

$$\begin{aligned} (8, Q, \heartsuit) &\leftrightarrow \{ 8\spadesuit, 8\diamondsuit, 8\heartsuit, 8\clubsuit, Q\heartsuit \} \\ (2, A, \clubsuit) &\leftrightarrow \{ 2\clubsuit, 2\heartsuit, 2\diamondsuit, 2\spadesuit, A\clubsuit \} \end{aligned}$$

Now we need only count the sequences. There are 13 ways to choose the first rank, 12 ways to choose the second rank, and 4 ways to choose the suit. Thus, by the Generalized Product Rule, there are $13 \cdot 12 \cdot 4 = 624$ hands with a Four-of-a-Kind. This means that only 1 hand in about 4165 has a Four-of-a-Kind; not surprisingly, this is considered a very good poker hand!

16.9.2 Hands with a Full House

A *Full House* is a hand with three cards of one rank and two cards of another rank. Here are some examples:

$$\begin{aligned} &\{ 2\spadesuit, 2\clubsuit, 2\diamondsuit, J\clubsuit, J\diamondsuit \} \\ &\{ 5\diamondsuit, 5\clubsuit, 5\heartsuit, 7\heartsuit, 7\clubsuit \} \end{aligned}$$

Again, we shift to a problem about sequences. There is a bijection between Full Houses and sequences specifying:

1. The rank of the triple, which can be chosen in 13 ways.
2. The suits of the triple, which can be selected in $\binom{4}{3}$ ways.
3. The rank of the pair, which can be chosen in 12 ways.
4. The suits of the pair, which can be selected in $\binom{4}{2}$ ways.

The example hands correspond to sequences as shown below:

$$\begin{aligned} (2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) &\leftrightarrow \{ 2\spadesuit, 2\clubsuit, 2\diamondsuit, J\clubsuit, J\diamondsuit \} \\ (5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) &\leftrightarrow \{ 5\diamondsuit, 5\clubsuit, 5\heartsuit, 7\heartsuit, 7\clubsuit \} \end{aligned}$$

By the Generalized Product Rule, the number of Full Houses is:

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$$

We're on a roll—but we're about to hit a speedbump.

16.9.3 Hands with Two Pairs

How many hands have *Two Pairs*; that is, two cards of one rank, two cards of another rank, and one card of a third rank? Here are examples:

$$\begin{aligned} & \{ 3\heartsuit, 3\spadesuit, Q\diamondsuit, Q\heartsuit, A\clubsuit \} \\ & \{ 9\heartsuit, 9\diamondsuit, 5\heartsuit, 5\clubsuit, K\spadesuit \} \end{aligned}$$

Each hand with Two Pairs is described by a sequence consisting of:

1. The rank of the first pair, which can be chosen in 13 ways.
2. The suits of the first pair, which can be selected $\binom{4}{2}$ ways.
3. The rank of the second pair, which can be chosen in 12 ways.
4. The suits of the second pair, which can be selected in $\binom{4}{2}$ ways.
5. The rank of the extra card, which can be chosen in 11 ways.
6. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

Thus, it might appear that the number of hands with Two Pairs is:

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4$$

Wrong answer! The problem is that there is *not* a bijection from such sequences to hands with Two Pairs. This is actually a 2-to-1 mapping. For example, here are the pairs of sequences that map to the hands given above:

$$\begin{aligned} (3, \{\heartsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit) & \searrow \\ (Q, \{\diamondsuit, \heartsuit\}, 3, \{\heartsuit, \spadesuit\}, A, \clubsuit) & \nearrow \\ & \{ 3\heartsuit, 3\spadesuit, Q\diamondsuit, Q\heartsuit, A\clubsuit \} \\ \\ (9, \{\heartsuit, \diamondsuit\}, 5, \{\heartsuit, \clubsuit\}, K, \spadesuit) & \searrow \\ (5, \{\heartsuit, \clubsuit\}, 9, \{\heartsuit, \diamondsuit\}, K, \spadesuit) & \nearrow \\ & \{ 9\heartsuit, 9\diamondsuit, 5\heartsuit, 5\clubsuit, K\spadesuit \} \end{aligned}$$

The problem is that nothing distinguishes the first pair from the second. A pair of 5's and a pair of 9's is the same as a pair of 9's and a pair of 5's. We avoided this difficulty in counting Full Houses because, for example, a pair of 6's and a triple of kings is different from a pair of kings and a triple of 6's.

We ran into precisely this difficulty last time, when we went from counting arrangements of *different* pieces on a chessboard to counting arrangements of two *identical* rooks. The solution then was to apply the Division Rule, and we can do the same here. In this case, the Division rule says there are twice as many sequences as hands, so the number of hands with Two Pairs is actually:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4}{2}$$

Another Approach

The preceding example was disturbing! One could easily overlook the fact that the mapping was 2-to-1 on an exam, fail the course, and turn to a life of crime. You can make the world a safer place in two ways:

1. Whenever you use a mapping $f : A \rightarrow B$ to translate one counting problem to another, check that the same number elements in A are mapped to each element in B . If k elements of A map to each of element of B , then apply the Division Rule using the constant k .
2. As an extra check, try solving the same problem in a different way. Multiple approaches are often available— and all had better give the same answer! (Sometimes different approaches give answers that *look* different, but turn out to be the same after some algebra.)

We already used the first method; let's try the second. There is a bijection between hands with two pairs and sequences that specify:

1. The ranks of the two pairs, which can be chosen in $\binom{13}{2}$ ways.
2. The suits of the lower-rank pair, which can be selected in $\binom{4}{2}$ ways.
3. The suits of the higher-rank pair, which can be selected in $\binom{4}{2}$ ways.
4. The rank of the extra card, which can be chosen in 11 ways.
5. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

For example, the following sequences and hands correspond:

$$\begin{aligned} (\{3, Q\}, \{\diamond, \spadesuit\}, \{\diamond, \heartsuit\}, A, \clubsuit) &\leftrightarrow \{ 3\diamond, 3\spadesuit, Q\diamond, Q\heartsuit, A\clubsuit \} \\ (\{9, 5\}, \{\heartsuit, \clubsuit\}, \{\heartsuit, \diamond\}, K, \spadesuit) &\leftrightarrow \{ 9\heartsuit, 9\diamond, 5\heartsuit, 5\clubsuit, K\spadesuit \} \end{aligned}$$

Thus, the number of hands with two pairs is:

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 11 \cdot 4$$

This is the same answer we got before, though in a slightly different form.

16.9.4 Hands with Every Suit

How many hands contain at least one card from every suit? Here is an example of such a hand:

$$\{ 7\diamond, K\clubsuit, 3\diamond, A\heartsuit, 2\spadesuit \}$$

Each such hand is described by a sequence that specifies:

1. The ranks of the diamond, the club, the heart, and the spade, which can be selected in $13 \cdot 13 \cdot 13 \cdot 13 = 13^4$ ways.

2. The suit of the extra card, which can be selected in 4 ways.
3. The rank of the extra card, which can be selected in 12 ways.

For example, the hand above is described by the sequence:

$$(7, K, A, 2, \diamond, 3) \leftrightarrow \{ 7\diamond, K\clubsuit, A\heartsuit, 2\spadesuit, 3\diamond \}$$

Are there other sequences that correspond to the same hand? There is one more! We could equally well regard either the $3\diamond$ or the $7\diamond$ as the extra card, so this is actually a 2-to-1 mapping. Here are the two sequences corresponding to the example hand:

$$\begin{array}{l} (7, K, A, 2, \diamond, 3) \searrow \\ (3, K, A, 2, \diamond, 7) \nearrow \end{array} \{ 7\diamond, K\clubsuit, A\heartsuit, 2\spadesuit, 3\diamond \}$$

Therefore, the number of hands with every suit is:

$$\frac{13^4 \cdot 4 \cdot 12}{2}$$

16.9.5 Problems

Class Problems

Problem 16.19.

Solve the following counting problems by defining an appropriate mapping (bijective or k -to-1) between a set whose size you know and the set in question.

(a) How many different ways are there to select a dozen donuts if four varieties are available?

(b) In how many ways can Mr. and Mrs. Grumperson distribute 13 identical pieces of coal to their two —no, three! —children for Christmas?

(c) How many solutions over the nonnegative integers are there to the inequality:

$$x_1 + x_2 + \dots + x_{10} \leq 100$$

(d) We want to count step-by-step paths between points in the plane with integer coordinates. Only two kinds of step are allowed: a right-step which increments the x coordinate, and an up-step which increments the y coordinate.

- (i) How many paths are there from $(0, 0)$ to $(20, 30)$?
- (ii) How many paths are there from $(0, 0)$ to $(20, 30)$ that go through the point $(10, 10)$?

- (iii) How many paths are there from $(0, 0)$ to $(20, 30)$ that do *not* go through either of the points $(10, 10)$ and $(15, 20)$?

Hint: Let P be the set of paths from $(0, 0)$ to $(20, 30)$, N_1 be the paths in P that go through $(10, 10)$ and N_2 be the paths in P that go through $(15, 20)$.

Problem 16.20.

Solve the following counting problems. Define an appropriate mapping (bijective or k -to-1) between a set whose size you know and the set in question.

(a) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. Write a multinomial coefficient for the number of ways this can be done.

(b) Write a multinomial coefficient for the number of nonnegative integer solutions for the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8. \quad (16.5)$$

(c) How many nonnegative integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 17?

Exam Problems

Problem 16.21.

Here are the solutions to the next 10 problem parts, in no particular order.

$$n^m \quad m^n \quad \frac{n!}{(n-m)!} \quad \binom{n+m}{m} \quad \binom{n-1+m}{m} \quad \binom{n-1+m}{n} \quad 2^{mn}$$

- (a) How many solutions over the natural numbers are there to the inequality _____
 $x_1 + x_2 + \cdots + x_n \leq m$?
- (b) How many length m words can be formed from an n -letter alphabet, if no _____
 letter is used more than once?
- (c) How many length m words can be formed from an n -letter alphabet, if _____
 letters can be reused?

- (d) How many binary relations are there from set A to set B when $|A| = m$ _____ and $|B| = n$?
- (e) How many injections are there from set A to set B , where $|A| = m$ and _____ $|B| = n \geq m$?
- (f) How many ways are there to place a total of m distinguishable balls into _____ n distinguishable urns, with some urns possibly empty or with several balls?
- (g) How many ways are there to place a total of m indistinguishable balls into _____ n distinguishable urns, with some urns possibly empty or with several balls?
- (h) How many ways are there to put a total of m distinguishable balls into n _____ distinguishable urns with at most one ball in each urn?

16.10 Inclusion-Exclusion

How big is a union of sets? For example, suppose there are 60 math majors, 200 EECS majors, and 40 physics majors. How many students are there in these three departments? Let M be the set of math majors, E be the set of EECS majors, and P be the set of physics majors. In these terms, we're asking for $|M \cup E \cup P|$.

The Sum Rule says that the size of union of *disjoint* sets is the sum of their sizes:

$$|M \cup E \cup P| = |M| + |E| + |P| \quad (\text{if } M, E, \text{ and } P \text{ are disjoint})$$

However, the sets M , E , and P might *not* be disjoint. For example, there might be a student majoring in both math and physics. Such a student would be counted twice on the right side of this equation, once as an element of M and once as an element of P . Worse, there might be a triple-major⁴ counted *three* times on the right side!

Our last counting rule determines the size of a union of sets that are not necessarily disjoint. Before we state the rule, let's build some intuition by considering some easier special cases: unions of just two or three sets.

16.10.1 Union of Two Sets

For two sets, S_1 and S_2 , the *Inclusion-Exclusion Rule* is that the size of their union is:

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| \quad (16.6)$$

⁴... though not at MIT anymore.

Intuitively, each element of S_1 is accounted for in the first term, and each element of S_2 is accounted for in the second term. Elements in *both* S_1 and S_2 are counted *twice*— once in the first term and once in the second. This double-counting is corrected by the final term.

We can capture this double-counting idea in a precise way by decomposing the union of S_1 and S_2 into three disjoint sets, the elements in each set but not the other, and the elements in both:

$$S_1 \cup S_2 = (S_1 - S_2) \cup (S_2 - S_1) \cup (S_1 \cap S_2). \quad (16.7)$$

Similarly, we can decompose each of S_1 and S_2 into the elements exclusively in each set and the elements in both:

$$S_1 = (S_1 - S_2) \cup (S_1 \cap S_2), \quad (16.8)$$

$$S_2 = (S_2 - S_1) \cup (S_1 \cap S_2). \quad (16.9)$$

Now we have from (16.8) and (16.9)

$$\begin{aligned} |S_1| + |S_2| &= (|S_1 - S_2| + |S_1 \cap S_2|) + (|S_2 - S_1| + |S_1 \cap S_2|) \\ &= |S_1 - S_2| + |S_2 - S_1| + 2|S_1 \cap S_2|, \end{aligned} \quad (16.10)$$

which shows the double-counting of $S_1 \cap S_2$ in the sum. On the other hand, we have from (16.7)

$$|S_1 \cup S_2| = |S_1 - S_2| + |S_2 - S_1| + |S_1 \cap S_2|. \quad (16.11)$$

Subtracting (16.11) from (16.10), we get

$$(|S_1| + |S_2|) - |S_1 \cup S_2| = |S_1 \cap S_2|$$

which proves (16.6).

16.10.2 Union of Three Sets

So how many students are there in the math, EECS, and physics departments? In other words, what is $|M \cup E \cup P|$ if:

$$|M| = 60$$

$$|E| = 200$$

$$|P| = 40$$

The size of a union of three sets is given by a more complicated Inclusion-Exclusion formula:

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| &= |S_1| + |S_2| + |S_3| \\ &\quad - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\ &\quad + |S_1 \cap S_2 \cap S_3| \end{aligned}$$

Remarkably, the expression on the right accounts for each element in the union of S_1 , S_2 , and S_3 exactly once. For example, suppose that x is an element of all three sets. Then x is counted three times (by the $|S_1|$, $|S_2|$, and $|S_3|$ terms), subtracted off three times (by the $|S_1 \cap S_2|$, $|S_1 \cap S_3|$, and $|S_2 \cap S_3|$ terms), and then counted once more (by the $|S_1 \cap S_2 \cap S_3|$ term). The net effect is that x is counted just once.

So we can't answer the original question without knowing the sizes of the various intersections. Let's suppose that there are:

- 4 math - EECS double majors
- 3 math - physics double majors
- 11 EECS - physics double majors
- 2 triple majors

Then $|M \cap E| = 4 + 2$, $|M \cap P| = 3 + 2$, $|E \cap P| = 11 + 2$, and $|M \cap E \cap P| = 2$. Plugging all this into the formula gives:

$$\begin{aligned} |M \cup E \cup P| &= |M| + |E| + |P| - |M \cap E| - |M \cap P| - |E \cap P| + |M \cap E \cap P| \\ &= 60 + 200 + 40 - 6 - 5 - 13 + 2 \\ &= 278 \end{aligned}$$

Sequences with 42, 04, or 60

In how many permutations of the set $\{0, 1, 2, \dots, 9\}$ do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively? For example, none of these pairs appears in:

$$(7, 2, 9, 5, 4, 1, 3, 8, 0, 6)$$

The 06 at the end doesn't count; we need 60. On the other hand, both 04 and 60 appear consecutively in this permutation:

$$(7, 2, 5, \underline{6}, \underline{0}, \underline{4}, 3, 8, 1, 9)$$

Let P_{42} be the set of all permutations in which 42 appears; define P_{60} and P_{04} similarly. Thus, for example, the permutation above is contained in both P_{60} and P_{04} . In these terms, we're looking for the size of the set $P_{42} \cup P_{04} \cup P_{60}$.

First, we must determine the sizes of the individual sets, such as P_{60} . We can use a trick: group the 6 and 0 together as a single symbol. Then there is a natural bijection between permutations of $\{0, 1, 2, \dots, 9\}$ containing 6 and 0 consecutively and permutations of:

$$\{60, 1, 2, 3, 4, 5, 7, 8, 9\}$$

For example, the following two sequences correspond:

$$(7, 2, 5, \underline{6}, \underline{0}, 4, 3, 8, 1, 9) \quad \leftrightarrow \quad (7, 2, 5, \underline{60}, 4, 3, 8, 1, 9)$$

There are $9!$ permutations of the set containing 60, so $|P_{60}| = 9!$ by the Bijection Rule. Similarly, $|P_{04}| = |P_{42}| = 9!$ as well.

Next, we must determine the sizes of the two-way intersections, such as $P_{42} \cap P_{60}$. Using the grouping trick again, there is a bijection with permutations of the set:

$$\{42, 60, 1, 3, 5, 7, 8, 9\}$$

Thus, $|P_{42} \cap P_{60}| = 8!$. Similarly, $|P_{60} \cap P_{04}| = 8!$ by a bijection with the set:

$$\{604, 1, 2, 3, 5, 7, 8, 9\}$$

And $|P_{42} \cap P_{04}| = 8!$ as well by a similar argument. Finally, note that $|P_{60} \cap P_{04} \cap P_{42}| = 7!$ by a bijection with the set:

$$\{6042, 1, 3, 5, 7, 8, 9\}$$

Plugging all this into the formula gives:

$$|P_{42} \cup P_{04} \cup P_{60}| = 9! + 9! + 9! - 8! - 8! - 8! + 7!$$

16.10.3 Union of n Sets

The size of a union of n sets is given by the following rule.

Rule 10 (Inclusion-Exclusion).

$$|S_1 \cup S_2 \cup \cdots \cup S_n| =$$

the sum of the sizes of the individual sets
 minus *the sizes of all two-way intersections*
 plus *the sizes of all three-way intersections*
 minus *the sizes of all four-way intersections*
 plus *the sizes of all five-way intersections, etc.*

The formulas for unions of two and three sets are special cases of this general rule.

This way of expressing Inclusion-Exclusion is easy to understand and nearly as precise as expressing it in mathematical symbols, but we'll need the symbolic version below, so let's work on deciphering it now.

We already have a standard notation for the sum of sizes of the individual sets, namely,

$$\sum_{i=1}^n |S_i|.$$

A "two-way intersection" is a set of the form $S_i \cap S_j$ for $i \neq j$. We regard $S_j \cap S_i$ as the same two-way intersection as $S_i \cap S_j$, so we can assume that $i < j$. Now we can express the sum of the sizes of the two-way intersections as

$$\sum_{1 \leq i < j \leq n} |S_i \cap S_j|.$$

Similarly, the sum of the sizes of the three-way intersections is

$$\sum_{1 \leq i < j < k \leq n} |S_i \cap S_j \cap S_k|.$$

These sums have alternating signs in the Inclusion-Exclusion formula, with the sum of the k -way intersections getting the sign $(-1)^{k-1}$. This finally leads to a symbolic version of the rule:

Rule (Inclusion-Exclusion).

$$\begin{aligned} \left| \bigcup_{i=1}^n S_i \right| &= \sum_{i=1}^n |S_i| \\ &\quad - \sum_{1 \leq i < j \leq n} |S_i \cap S_j| \\ &\quad + \sum_{1 \leq i < j < k \leq n} |S_i \cap S_j \cap S_k| + \cdots \\ &\quad + (-1)^{n-1} \left| \bigcap_{i=1}^n S_i \right|. \end{aligned}$$

16.10.4 Computing Euler's Function

We will now use Inclusion-Exclusion to calculate Euler's function, $\phi(n)$. By definition, $\phi(n)$ is the number of nonnegative integers less than a positive integer n that are relatively prime to n . But the set, S , of nonnegative integers less than n that are *not* relatively prime to n will be easier to count.

Suppose the prime factorization of n is $p_1^{e_1} \cdots p_m^{e_m}$ for distinct primes p_i . This means that the integers in S are precisely the nonnegative integers less than n that are divisible by at least one of the p_i 's. So, letting C_i be the set of nonnegative integers less than n that are divisible by p_i , we have

$$S = \bigcup_{i=1}^m C_i.$$

We'll be able to find the size of this union using Inclusion-Exclusion because the intersections of the C_i 's are easy to count. For example, $C_1 \cap C_2 \cap C_3$ is the set of nonnegative integers less than n that are divisible by each of p_1 , p_2 and p_3 . But since the p_i 's are distinct primes, being divisible by each of these primes is the same as being divisible by their product. Now observe that if r is a positive divisor of n , then exactly n/r nonnegative integers less than n are divisible by r , namely, $0, r, 2r, \dots, ((n/r) - 1)r$. So exactly $n/p_1 p_2 p_3$ nonnegative integers less than n are divisible by all three primes p_1, p_2, p_3 . In other words,

$$|C_1 \cap C_2 \cap C_3| = \frac{n}{p_1 p_2 p_3}.$$

So reasoning this way about all the intersections among the C_i 's and applying Inclusion-Exclusion, we get

$$\begin{aligned}
 |S| &= \left| \bigcup_{i=1}^m C_i \right| \\
 &= \sum_{i=1}^m |C_i| - \sum_{1 \leq i < j \leq m} |C_i \cap C_j| + \sum_{1 \leq i < j < k \leq m} |C_i \cap C_j \cap C_k| - \cdots + (-1)^{m-1} \left| \bigcap_{i=1}^m C_i \right| \\
 &= \sum_{i=1}^m \frac{n}{p_i} - \sum_{1 \leq i < j \leq m} \frac{n}{p_i p_j} + \sum_{1 \leq i < j < k \leq m} \frac{n}{p_i p_j p_k} - \cdots + (-1)^{m-1} \frac{n}{p_1 p_2 \cdots p_m} \\
 &= n \left(\sum_{i=1}^m \frac{1}{p_i} - \sum_{1 \leq i < j \leq m} \frac{1}{p_i p_j} + \sum_{1 \leq i < j < k \leq m} \frac{1}{p_i p_j p_k} - \cdots + (-1)^{m-1} \frac{1}{p_1 p_2 \cdots p_m} \right)
 \end{aligned}$$

But $\phi(n) = n - |S|$ by definition, so

$$\begin{aligned}
 \phi(n) &= n \left(1 - \sum_{i=1}^m \frac{1}{p_i} + \sum_{1 \leq i < j \leq m} \frac{1}{p_i p_j} - \sum_{1 \leq i < j < k \leq m} \frac{1}{p_i p_j p_k} + \cdots + (-1)^m \frac{1}{p_1 p_2 \cdots p_m} \right) \\
 &= n \prod_{i=1}^m \left(1 - \frac{1}{p_i} \right). \tag{16.12}
 \end{aligned}$$

Notice that in case $n = p^k$ for some prime, p , then (16.12) simplifies to

$$\phi(p^k) = p^k \left(1 - \frac{1}{p} \right) = p^k - p^{k-1}$$

as claimed in chapter 14.

Quick Question: Why does equation (16.12) imply that

$$\phi(ab) = \phi(a)\phi(b)$$

for relatively prime integers $a, b > 1$, as claimed in Theorem 14.7.1.(a)?

16.10.5 Problems

Practice Problems

Problem 16.22.

The working days in the next year can be numbered 1, 2, 3, ..., 300. I'd like to avoid as many as possible.

- On even-numbered days, I'll say I'm sick.
- On days that are a multiple of 3, I'll say I was stuck in traffic.

- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I *avoid* in the coming year?

Class Problems

Problem 16.23.

A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password will be a cword which does not contain any of the subwords "fails", "failed", or "drop".

For example, the following two words are passwords:

adefiloprs, srpolifeda,

but the following three cwords are not:

adropeflis, failedrops, dropefails.

- How many cwords contain the subword "drop"?
- How many cwords contain both "drop" and "fails"?
- Use the Inclusion-Exclusion Principle to find a simple formula for the number of passwords.

Homework Problems

Problem 16.24.

How many paths are there from point $(0, 0)$ to $(50, 50)$ if every step increments one coordinate and leaves the other unchanged? How many are there when there are impassable boulders sitting at points $(10, 11)$ and $(21, 20)$? (You do not have to calculate the number explicitly; your answer may be an expression involving binomial coefficients.)

Hint: Count the number of paths going through $(10, 11)$, the number through $(21, 20)$, and use Inclusion-Exclusion.

Problem 16.25.

A *derangement* is a permutation (x_1, x_2, \dots, x_n) of the set $\{1, 2, \dots, n\}$ such that $x_i \neq i$ for all i . For example, $(2, 3, 4, 5, 1)$ is a derangement, but $(2, 1, 3, 5, 4)$ is not because 3 appears in the third position. The objective of this problem is to count derangements.

It turns out to be easier to start by counting the permutations that are *not* derangements. Let S_i be the set of all permutations (x_1, x_2, \dots, x_n) that are not derangements because $x_i = i$. So the set of non-derangements is

$$\bigcup_{i=1}^n S_i.$$

- (a) What is $|S_i|$?
- (b) What is $|S_i \cap S_j|$ where $i \neq j$?
- (c) What is $|S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_k}|$ where i_1, i_2, \dots, i_k are all distinct?
- (d) Use the inclusion-exclusion formula to express the number of non-derangements in terms of sizes of possible intersections of the sets S_1, \dots, S_n .
- (e) How many terms in the expression in part (d) have the form $|S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_k}|$?
- (f) Combine your answers to the preceding parts to prove the number of non-derangements is:

$$n! \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots \pm \frac{1}{n!} \right).$$

Conclude that the number of derangements is

$$n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \pm \frac{1}{n!} \right).$$

(g) As n goes to infinity, the number of derangements approaches a constant fraction of all permutations. What is that constant? *Hint:*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Problem 16.26.

How many of the numbers $2, \dots, n$ are prime? The Inclusion-Exclusion Principle offers a useful way to calculate the answer when n is large. Actually, we will use Inclusion-Exclusion to count the number of *composite* (nonprime) integers from 2 to n . Subtracting this from $n - 1$ gives the number of primes.

Let C_n be the set of composites from 2 to n , and let A_m be the set of numbers in the range $m + 1, \dots, n$ that are divisible by m . Notice that by definition, $A_m = \emptyset$ for $m \geq n$. So

$$C_n = \bigcup_{i=2}^{n-1} A_i. \quad (16.13)$$

- (a) Verify that if $m \mid k$, then $A_m \supseteq A_k$.

(b) Explain why the right hand side of (16.13) equals

$$\bigcup_{\text{primes } p \leq \sqrt{n}} A_p. \quad (16.14)$$

(c) Explain why $|A_m| = \lfloor n/m \rfloor - 1$ for $m \geq 2$.

(d) Consider any two relatively prime numbers $p, q \leq n$. What is the one number in $(A_p \cap A_q) - A_{p \cdot q}$?

(e) Let \mathcal{P} be a finite set of at least two primes. Give a simple formula for

$$\left| \bigcap_{p \in \mathcal{P}} A_p \right|.$$

(f) Use the Inclusion-Exclusion principle to obtain a formula for $|C_{150}|$ in terms of the sizes of intersections among the sets $A_2, A_3, A_5, A_7, A_{11}$. (Omit the intersections that are empty; for example, any intersection of more than three of these sets must be empty.)

(g) Use this formula to find the number of primes up to 150.

16.11 Binomial Theorem

Counting gives insight into one of the basic theorems of algebra. A *binomial* is a sum of two terms, such as $a + b$. Now consider its 4th power, $(a + b)^4$.

If we multiply out this 4th power expression completely, we get

$$\begin{aligned} (a + b)^4 = & \quad aaaa + aaab + aaba + aabb \\ & + abaa + abab + abba + abbb \\ & + baaa + baab + baba + babb \\ & + bbaa + bbab + bbba + bbbb \end{aligned}$$

Notice that there is one term for every sequence of a 's and b 's. So there are 2^4 terms, and the number of terms with k copies of b and $n - k$ copies of a is:

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

by the Bookkeeper Rule. Now let's group equivalent terms, such as $aaab = aaba = abaa = baaa$. Then the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. So for $n = 4$, this means:

$$(a + b)^4 = \binom{4}{0} \cdot a^4b^0 + \binom{4}{1} \cdot a^3b^1 + \binom{4}{2} \cdot a^2b^2 + \binom{4}{3} \cdot a^1b^3 + \binom{4}{4} \cdot a^0b^4$$

In general, this reasoning gives the Binomial Theorem:

Theorem 16.11.1 (Binomial Theorem). For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

The expression $\binom{n}{k}$ is often called a “binomial coefficient” in honor of its appearance here.

This reasoning about binomials extends nicely to *multinomials*, which are sums of two or more terms. For example, suppose we wanted the coefficient of

$$bo^2k^2e^3pr$$

in the expansion of $(b + o + k + e + p + r)^{10}$. Each term in this expansion is a product of 10 variables where each variable is one of $b, o, k, e, p,$ or r . Now, the coefficient of $bo^2k^2e^3pr$ is the number of those terms with exactly 1 b , 2 o 's, 2 k 's, 3 e 's, 1 p , and 1 r . And the number of such terms is precisely the number of rearrangements of the word BOOKKEEPER:

$$\binom{10}{1, 2, 2, 3, 1, 1} = \frac{10!}{1! 2! 2! 3! 1! 1!}.$$

The expression on the left is called a “multinomial coefficient.” This reasoning extends to a general theorem.

Definition 16.11.2. For $n, k_1, \dots, k_m \in \text{natural numbers}$, such that $k_1 + k_2 + \dots + k_m = n$, define the *multinomial coefficient*

$$\binom{n}{k_1, k_2, \dots, k_m} ::= \frac{n!}{k_1! k_2! \dots k_m!}.$$

Theorem 16.11.3 (Multinomial Theorem). For all $n \in \mathbb{N}$ and $z_1, \dots, z_m \in \mathbb{R}$:

$$(z_1 + z_2 + \dots + z_m)^n = \sum_{\substack{k_1, \dots, k_m \in \mathbb{N} \\ k_1 + \dots + k_m = n}} \binom{n}{k_1, k_2, \dots, k_m} z_1^{k_1} z_2^{k_2} \dots z_m^{k_m}$$

You'll be better off remembering the reasoning behind the Multinomial Theorem rather than this ugly formal statement.

16.11.1 Problems

Practice Problems

Problem 16.27.

Find the coefficients of $x^{10}y^5$ in $(19x + 4y)^{15}$

Class Problems**Problem 16.28.**

Find the coefficients of

- (a) x^5 in $(1 + x)^{11}$
- (b) x^8y^9 in $(3x + 2y)^{17}$
- (c) a^6b^6 in $(a^2 + b^3)^5$

Problem 16.29. (a) Use the Multinomial Theorem 16.11.3 to prove that

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p} \quad (16.15)$$

for all primes p . (Do not prove it using Fermat's "little" Theorem. The point of this problem is to offer an independent proof of Fermat's theorem.)

Hint: Explain why $\binom{p}{k_1, k_2, \dots, k_n}$ is divisible by p if all the k_i 's are positive integers less than p .

(b) Explain how (16.15) immediately proves Fermat's Little Theorem 14.6.4: $n^{p-1} \equiv 1 \pmod{p}$ when n is not a multiple of p .

Homework Problems**Problem 16.30.**

The *degree sequence* of a simple graph is the weakly decreasing sequence of degrees of its vertices. For example, the degree sequence for the 5-vertex numbered tree pictured in the Figure 16.2 is $(2, 2, 2, 1, 1)$ and for the 7-vertex tree it is $(3, 3, 2, 1, 1, 1, 1)$.

We're interested in counting how many numbered trees there are with a given degree sequence. We'll do this using the bijection defined in Problem 16.5 between n -vertex numbered trees and length $n-2$ code words whose characters are integers between 1 and n .

The *occurrence number* for a character in a word is the number of times that the character occurs in the word. For example, in the word 65622, the occurrence number for 6 is two, and the occurrence number for 5 is one. The *occurrence sequence* of a word is the weakly decreasing sequence of occurrence numbers of characters in the word. The occurrence sequence for this word is $(2, 2, 1)$ because it has two occurrences of each of the characters 6 and 2, and one occurrence of 5.

(a) There is simple relationship between the degree sequence of an n -vertex numbered tree and the occurrence sequence of its code. Describe this relationship and explain why it holds. Conclude that counting n -vertex numbered trees with a given degree sequence is the same as counting the number of length $n-2$ code words with a given occurrence sequence.

Hint: How many times does a vertex of degree, d , occur in the code?

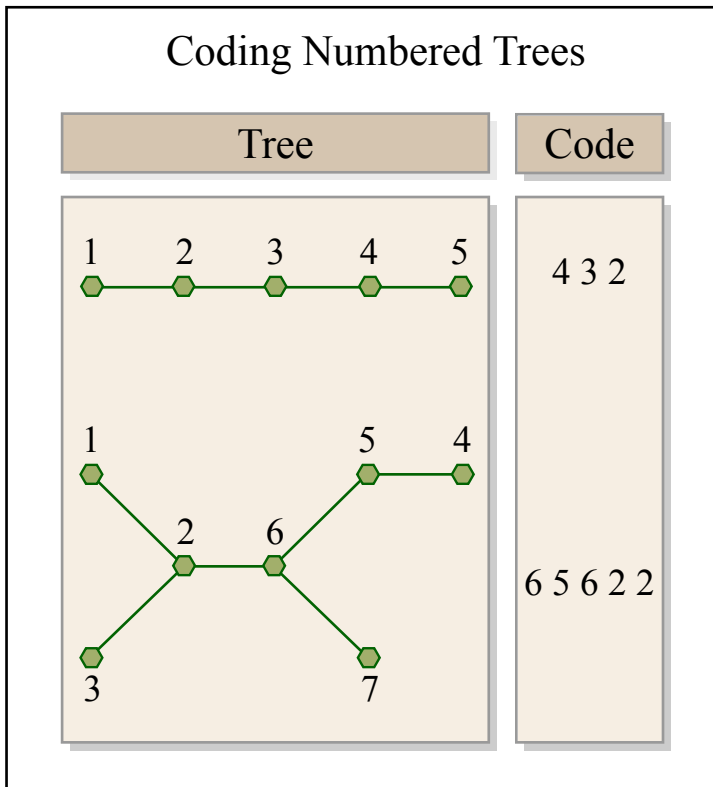


Image by MIT OpenCourseWare.

Figure 16.2:

For simplicity, let's focus on counting 9-vertex numbered trees with a given degree sequence. By part (a), this is the same as counting the number of length 7 code words with a given occurrence sequence.

Any length 7 code word has a *pattern*, which is another length 7 word over the alphabet a, b, c, d, e, f, g that has the same occurrence sequence.

(b) How many length 7 patterns are there with three occurrences of a , two occurrences of b , and one occurrence of c and d ?

(c) How many ways are there to assign occurrence numbers to integers $1, 2, \dots, 9$ so that a code word with those occurrence numbers would have the occurrence sequence $3, 2, 1, 1, 0, 0, 0, 0, 0$?

In general, to find the pattern of a code word, list its characters in decreasing order by *number of occurrences*, and list characters with the same number of occurrences in decreasing order. Then replace successive characters in the list by successive letters a, b, c, d, e, f, g . The code word 2468751 , for example, has the pattern $fecabdg$, which is obtained by replacing its characters $8, 7, 6, 5, 4, 2, 1$ by a, b, c, d, e, f, g , respectively. The code word 2449249 has pattern $caabcab$, which is obtained by replacing its characters $4, 9, 2$ by a, b, c , respectively.

(d) What length 7 code word has three occurrences of 7, two occurrences of 8, one occurrence each of 2 and 9, and pattern abacbad?

(e) Explain why the number of 9-vertex numbered trees with degree sequence $(4, 3, 2, 2, 1, 1, 1, 1, 1)$ is the product of the answers to parts (b) and (c).

16.12 Combinatorial Proof

Suppose you have n different T-shirts, but only want to keep k . You could equally well select the k shirts you want to keep or select the complementary set of $n - k$ shirts you want to throw out. Thus, the number of ways to select k shirts from among n must be equal to the number of ways to select $n - k$ shirts from among n . Therefore:

$$\binom{n}{k} = \binom{n}{n-k}$$

This is easy to prove algebraically, since both sides are equal to:

$$\frac{n!}{k!(n-k)!}$$

But we didn't really have to resort to algebra; we just used counting principles. Hmm.

16.12.1 Boxing

Jay, famed 6.042 TA, has decided to try out for the US Olympic boxing team. After all, he's watched all of the *Rocky* movies and spent hours in front of a mirror sneering, "Yo, you wanna piece a' me?!" Jay figures that n people (including himself) are competing for spots on the team and only k will be selected. As part of maneuvering for a spot on the team, he needs to work out how many different teams are possible. There are two cases to consider:

- Jay *is* selected for the team, and his $k - 1$ teammates are selected from among the other $n - 1$ competitors. The number of different teams that can be formed in this way is:

$$\binom{n-1}{k-1}$$

- Jay is *not* selected for the team, and all k team members are selected from among the other $n - 1$ competitors. The number of teams that can be formed this way is:

$$\binom{n-1}{k}$$

All teams of the first type contain Jay, and no team of the second type does; therefore, the two sets of teams are disjoint. Thus, by the Sum Rule, the total number of possible Olympic boxing teams is:

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

Jeremy, equally-famed 6.042 TA, thinks Jay isn't so tough and so he might as well also try out. He reasons that n people (including himself) are trying out for k spots. Thus, the number of ways to select the team is simply:

$$\binom{n}{k}$$

Jeremy and Jay each correctly counted the number of possible boxing teams; thus, their answers must be equal. So we know:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

This is called *Pascal's Identity*. And we proved it *without any algebra!* Instead, we relied purely on counting techniques.

16.12.2 Finding a Combinatorial Proof

A *combinatorial proof* is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S .
2. Show that $|S| = n$ by counting one way.
3. Show that $|S| = m$ by counting another way.
4. Conclude that $n = m$.

In the preceding example, S was the set of all possible Olympic boxing teams. Jay computed

$$|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

by counting one way, and Jeremy computed

$$|S| = \binom{n}{k}$$

by counting another. Equating these two expressions gave Pascal's Identity.

More typically, the set S is defined in terms of simple sequences or sets rather than an elaborate story. Here is less colorful example of a combinatorial argument.

Theorem 16.12.1.

$$\sum_{r=0}^n \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n}$$

Proof. We give a combinatorial proof. Let S be all n -card hands that can be dealt from a deck containing n red cards (numbered $1, \dots, n$) and $2n$ black cards (numbered $1, \dots, 2n$). First, note that every $3n$ -element set has

$$|S| = \binom{3n}{n}$$

n -element subsets.

From another perspective, the number of hands with exactly r red cards is

$$\binom{n}{r} \binom{2n}{n-r}$$

since there are $\binom{n}{r}$ ways to choose the r red cards and $\binom{2n}{n-r}$ ways to choose the $n-r$ black cards. Since the number of red cards can be anywhere from 0 to n , the total number of n -card hands is:

$$|S| = \sum_{r=0}^n \binom{n}{r} \binom{2n}{n-r}$$

Equating these two expressions for $|S|$ proves the theorem. ■

Combinatorial proofs are almost magical. Theorem 16.12.1 looks pretty scary, but we proved it without any algebraic manipulations at all. The key to constructing a combinatorial proof is choosing the set S properly, which can be tricky. Generally, the simpler side of the equation should provide some guidance. For example, the right side of Theorem 16.12.1 is $\binom{3n}{n}$, which suggests choosing S to be all n -element subsets of some $3n$ -element set.

16.12.3 Problems**Class Problems****Problem 16.31.**

According to the Multinomial theorem, $(w + x + y + z)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, r_3, r_4} w^{r_1} x^{r_2} y^{r_3} z^{r_4}.$$

(a) How many terms are there in the sum?

(b) The sum of these multinomial coefficients has an easily expressed value. What is it?

$$\sum_{\substack{r_1+r_2+r_3+r_4=n, \\ r_i \in \mathbb{N}}} \binom{n}{r_1, r_2, r_3, r_4} =? \tag{16.16}$$

Hint: How many terms are there when $(w + x + y + z)^n$ is expressed as a sum of monomials in w, x, y, z before terms with like powers of these variables are collected together under a single coefficient?

Homework Problems

Problem 16.32.

Prove the following identity by algebraic manipulation and by giving a combinatorial argument:

$$\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}$$

Problem 16.33. (a) Find a combinatorial (*not* algebraic) proof that

$$\sum_{i=0}^n \binom{n}{i} = 2^n.$$

(b) Below is a combinatorial proof of an equation. What is the equation?

Proof. Stinky Peterson owns n newts, t toads, and s slugs. Conveniently, he lives in a dorm with $n + t + s$ other students. (The students are distinguishable, but creatures of the same variety are not distinguishable.) Stinky wants to put one creature in each neighbor's bed. Let W be the set of all ways in which this can be done.

On one hand, he could first determine who gets the slugs. Then, he could decide who among his remaining neighbors has earned a toad. Therefore, $|W|$ is equal to the expression on the left.

On the other hand, Stinky could first decide which people deserve newts and slugs and then, from among those, determine who truly merits a newt. This shows that $|W|$ is equal to the expression on the right.

Since both expressions are equal to $|W|$, they must be equal to each other. ■

(Combinatorial proofs are real proofs. They are not only rigorous, but also convey an intuitive understanding that a purely algebraic argument might not reveal. However, combinatorial proofs are usually less colorful than this one.)

Problem 16.34.

According to the Multinomial Theorem 16.11.3, $(x_1 + x_2 + \dots + x_k)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, \dots, r_k} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}.$$

- (a) How many terms are there in the sum?
- (b) The sum of these multinomial coefficients has an easily expressed value:

$$\sum_{\substack{r_1+r_2+\dots+r_k=n, \\ r_i \in \mathbb{N}}} \binom{n}{r_1, r_2, \dots, r_k} = k^n \quad (16.17)$$

Give a combinatorial proof of this identity.

Hint: How many terms are there when $(x_1 + x_2 + \dots + x_k)^n$ is expressed as a sum of monomials in x_i before terms with like powers of these variables are collected together under a single coefficient?

Problem 16.35.

Give combinatorial proofs of the identities below. Use the following structure for each proof. First, define an appropriate set S . Next, show that the left side of the equation counts the number of elements in S . Then show that, from another perspective, the right side of the equation also counts the number of elements in set S . Conclude that the left side must be equal to the right, since both are equal to $|S|$.

(a)

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k} \cdot \binom{n}{n-k}$$

(b)

$$\sum_{i=0}^r \binom{n+i}{i} = \binom{n+r+1}{r}$$

Hint: consider a set of binary strings that could be counted using the right side of the equation, then try partitioning them into subsets countable by the elements of the sum on the left.

MIT OpenCourseWare
<http://ocw.mit.edu>

6.042J / 18.062J Mathematics for Computer Science
Spring 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.