Chapter 12

Planar Graphs

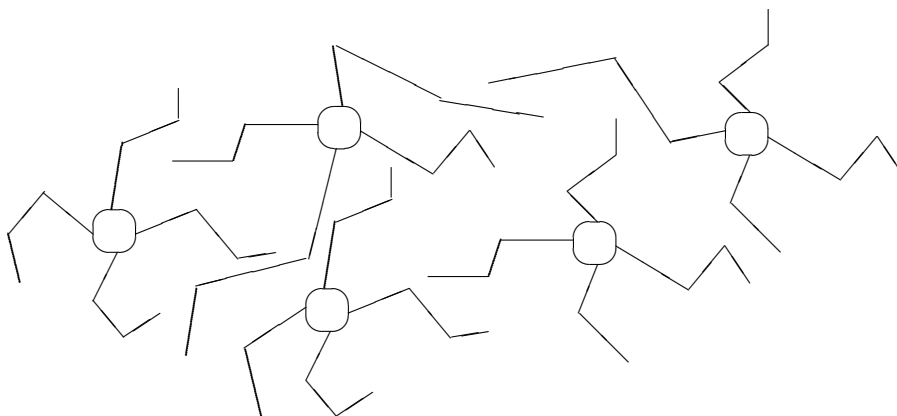## 12.1 Drawing Graphs in the Plane

Here are three dogs and three houses.

Dog   Dog   Dog

Can you find a path from each dog to each house such that no two paths intersect?

A *quadapus* is a little-known animal similar to an octopus, but with four arms. Here are five quadapi resting on the seafloor:

Can each quadapus simultaneously shake hands with every other in such a way that no arms cross?

Informally, a *planar graph* is a graph that can be drawn in the plane so that no edges cross, as in a map of showing the borders of countries or states. Thus, these two puzzles are asking whether the graphs below are planar; that is, whether they can be redrawn so that no edges cross. The first graph is called the *complete bipartite graph*, $K_{3,3}$, and the second is K_5.
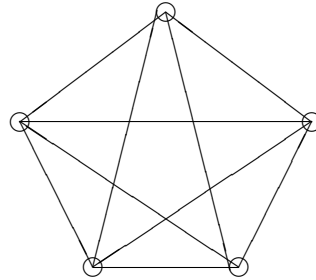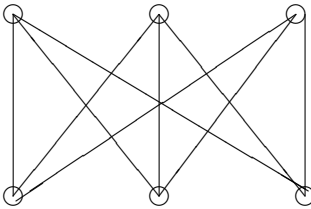
In each case, the answer is, "No— but almost!" In fact, each drawing *would* be possible if any single edge were removed.

Planar graphs have applications in circuit layout and are helpful in displaying graphical data, for example, program flow charts, organizational charts, and scheduling conflicts. We will treat them as a recursive data type and use structural induction to establish their basic properties. Then we'll be able to describe a simple recursive procedure to color any planar graph with *five* colors, and also prove that there is no uniform way to place n satellites around the globe unless $n = 4, 6, 8, 12,$ or 20.

When wires are arranged on a surface, like a circuit board or microchip, crossings require troublesome three-dimensional structures. When Steve Wozniak designed the disk drive for the early Apple II computer, he struggled mightly to achieve a nearly planar design:

> For two weeks, he worked late each night to make a satisfactory design. When he was finished, he found that if he moved a connector he could cut down on feedthroughs, making the board more reliable. To make that move, however, he had to start over in his design. This time it only took twenty hours. He then saw another feedthrough that could be eliminated, and again started over on his design. "The final design was generally recognized by computer engineers as brilliant and was by engineering aesthetics beautiful. Woz later said, 'It's something you can only do if you're the engineer and the PC board layout person yourself. That was an artistic layout. The board has virtually no feedthroughs.'"[a]

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[a]From apple2history.org which in turn quotes *Fire in the Valley* by Freiberger and Swaine.

12.2 Continuous & Discrete Faces

Planar graphs are graphs that can be drawn in the plane —like familiar maps of countries or states. "Drawing" the graph means that each vertex of the graph corresponds to a distinct point in the plane, and if two vertices are adjacent, their vertices are connected by a smooth, non-self-intersecting curve. None of the curves may "cross" —the only points that may appear on more than one curve are the vertex points. These curves are the boundaries of connected regions of the plane called the *continuous faces* of the drawing.

For example, the drawing in Figure 12.1 has four continuous faces. Face IV, which extends off to infinity in all directions, is called the *outside face*.

This definition of planar graphs is perfectly precise, but completely unsatisfying: it invokes smooth curves and continuous regions of the plane to define a property of a discrete data type. So the first thing we'd like to find is a discrete data type that represents planar drawings.

The clue to how to do this is to notice that the vertices along the boundary of each of the faces in Figure 12.1 form a simple cycle. For example, labeling the vertices as in Figure 12.2, the simple cycles for the face boundaries are

$$abca \qquad abda \qquad bcdb \qquad acda.$$

Since every edge in the drawing appears on the boundaries of exactly two continuous faces, every edge of the simple graph appears on exactly two of the simple cycles.

Vertices around the boundaries of states and countries in an ordinary map are
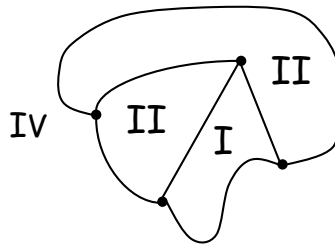
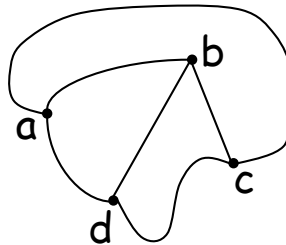Figure 12.1: A Planar Drawing with Four Faces.

Figure 12.2: The Drawing with Labelled Vertices.

always simple cycles, but oceans are slightly messier. The ocean boundary is the set of all boundaries of islands and continents in the ocean; it is a *set* of simple cycles (this can happen for countries too —like Bangladesh). But this happens because islands (and the two parts of Bangladesh) are not connected to each other. So we can dispose of this complication by treating each connected component separately.

But general planar graphs, even when they are connected, may be a bit more complicated than maps. For example a planar graph may have a "bridge," as in Figure 12.3. Now the cycle around the outer face is

<div align="center">

`abcefgecda.`

</div>

This is not a simple cycle, since it has to traverse the bridge c—e twice.

Planar graphs may also have "dongles," as in Figure 12.4. Now the cycle around the inner face is

<div align="center">

`rstvxyxvwvtur,`

</div>
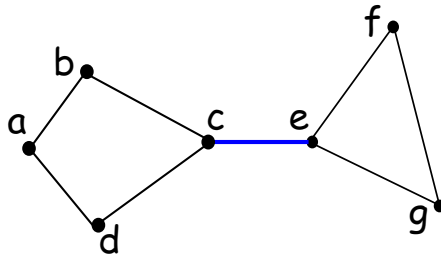
Figure 12.3: A Planar Drawing with a *Bridge*.
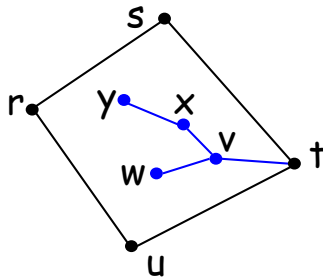
Figure 12.4: A Planar Drawing with a *Dongle*.

because it has to traverse *every* edge of the dongle twice —once "coming" and once "going."

But bridges and dongles are really the only complications, which leads us to the discrete data type of *planar embeddings* that we can use in place of continuous planar drawings. Namely, we'll define a planar embedding recursively to be the set of boundary-tracing cycles we could get drawing one edge after another.

12.3 Planar Embeddings

By thinking of the process of drawing a planar graph edge by edge, we can give a useful recursive definition of planar embeddings.

Definition 12.3.1. A *planar embedding* of a *connected* graph consists of a nonempty set of cycles of the graph called the *discrete faces* of the embedding. Planar embed-

dings are defined recursively as follows:

- **Base case:** If G is a graph consisting of a single vertex, v, then a planar embedding of G has one discrete face, namely the length zero cycle, v.

- **Constructor Case:** (split a face) Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, γ, of the planar embedding. That is, γ is a cycle of the form

$$a \ldots b \cdots a.$$

Then the graph obtained by adding the edge a—b to the edges of G has a planar embedding with the same discrete faces as G, except that face γ is replaced by the two discrete faces[1]

$$a \ldots ba \quad \text{and} \quad ab \cdots a,$$

as illustrated in Figure 12.5.

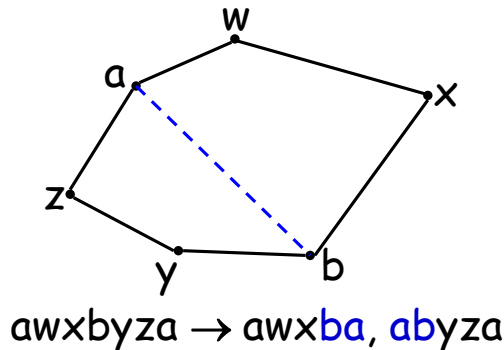$$awxbyza \rightarrow awxba,\ abyza$$

Figure 12.5: The Split a Face Case.

- **Constructor Case:** (add a bridge) Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Let a be a vertex on a discrete face, γ, in the embedding of G. That is, γ is of the form

$$a \ldots a.$$

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[1] There is one exception to this rule. If G is a line graph beginning with a and ending with b, then the cycles into which γ splits are actually the same. That's because adding edge a—b creates a simple cycle graph, C_n, that divides the plane into an "inner" and an "outer" region with the same border. In order to maintain the correspondence between continuous faces and discrete faces, we have to allow two "copies" of this same cycle to count as discrete faces. But since this is the only situation in which two faces are actually the same cycle, this exception is better explained in a footnote than mentioned explicitly in the definition.

Similarly, let b be a vertex on a discrete face, δ, in the embedding of H, so δ is of the form

$$b \cdots b.$$

Then the graph obtained by connecting G and H with a new edge, a—b, has a planar embedding whose discrete faces are the union of the discrete faces of G and H, except that faces γ and δ are replaced by one new face

$$a \ldots ab \cdots ba.$$

This is illustrated in Figure 12.6, where the faces of G and H are:

$$G : \{\texttt{axyza, axya, ayza}\} \qquad H : \{\texttt{btuvwb, btvwb, tuvt}\},$$

and after adding the bridge a—b, there is a single connected graph with faces

$$\{\texttt{axyzabtuvwba, axya, ayza, btvwb, tuvt}\}.$$

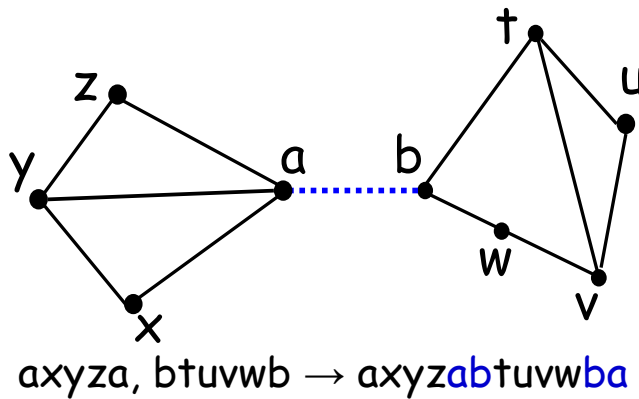
axyza, btuvwb → axyzabtuvwba

Figure 12.6: The Add Bridge Case.

An arbitrary graph is *planar* iff each of its connected components has a planar embedding.

12.4   What outer face?

Notice that the definition of planar embedding does not distinguish an "outer" face. There really isn't any need to distinguish one.

In fact, a planar embedding could be drawn with any given face on the outside. An intuitive explanation of this is to think of drawing the embedding on a *sphere* instead of the plane. Then any face can be made the outside face by "puncturing" that face of the sphere, stretching the puncture hole to a circle around the rest of the faces, and flattening the circular drawing onto the plane.

So pictures that show different "outside" boundaries may actually be illustrations of the same planar embedding.

This is what justifies the "add bridge" case in a planar embedding: whatever face is chosen in the embeddings of each of the disjoint planar graphs, we can draw a bridge between them without needing to cross any other edges in the drawing, because we can assume the bridge connects two "outer" faces.

12.5   Euler's Formula

The value of the recursive definition is that it provides a powerful technique for proving properties of planar graphs, namely, structural induction.

One of the most basic properties of a connected planar graph is that its number of vertices and edges determines the number of faces in every possible planar embedding:

Theorem 12.5.1 (Euler's Formula)**.** *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

For example, in Figure 12.1, $|V| = 4$, $|E| = 6$, and $f = 4$. Sure enough, $4-6+4 = 2$, as Euler's Formula claims.

Proof. The proof is by structural induction on the definition of planar embeddings. Let $P(\mathcal{E})$ be the proposition that $v - e + f = 2$ for an embedding, $\mathcal{E}$.

Base case: ($\mathcal{E}$ is the one vertex planar embedding). By definition, $v = 1$, $e = 0$, and $f = 1$, so $P(\mathcal{E})$ indeed holds.

Constructor case: (split a face) Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, $\gamma = a \ldots b \cdots a$, of the planar embedding.

Then the graph obtained by adding the edge a—b to the edges of G has a planar embedding with one more face and one more edge than G. So the quantity $v - e + f$ will remain the same for both graphs, and since by structural induction this quantity is 2 for G's embedding, it's also 2 for the embedding of G with the added edge. So P holds for the constructed embedding.

Constructor case: (add bridge) Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Then connecting these two graphs with a bridge merges the two bridged faces into a single face, and leaves all other faces unchanged. So the bridge operation yields a planar embedding of a connected graph with $v_G + v_H$ vertices, $e_G + e_H + 1$ edges, and $f_G + f_H - 1$ faces. But

$$
\begin{aligned}
&(v_G + v_H) - (e_G + e_H + 1) + (f_G + f_H - 1) \\
&= (v_G - e_G + f_G) + (v_H - e_H + f_H) - 2 \\
&= (2) + (2) - 2 \qquad\qquad\qquad \text{(by structural induction hypothesis)} \\
&= 2.
\end{aligned}
$$

So $v - e + f$ remains equal to 2 for the constructed embedding. That is, P also holds in this case.

This completes the proof of the constructor cases, and the theorem follows by structural induction. ∎

12.6 Number of Edges versus Vertices

Like Euler's formula, the following lemmas follow by structural induction directly from the definition of planar embedding.

Lemma 12.6.1. *In a planar embedding of a connected graph, each edge is traversed once by each of two different faces, or is traversed exactly twice by one face.*

Lemma 12.6.2. *In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.*

Corollary 12.6.3. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6.$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Lemma 12.6.1, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 12.6.2, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \qquad\qquad (12.1)$$

But $f = e - v + 2$ by Euler's formula, and substituting into (12.1) gives

$$
\begin{aligned}
3(e - v + 2) &\leq 2e \\
e - 3v + 6 &\leq 0 \\
e &\leq 3v - 6
\end{aligned}
$$

∎

Corollary 12.6.3 lets us prove that the quadapi can't all shake hands without crossing. Representing quadapi by vertices and the necessary handshakes by edges, we get the complete graph, K_5. Shaking hands without crossing amounts to showing that K_5 is planar. But K_5 is connected, has 5 vertices and 10 edges, and $10 > 3 \cdot 5 - 6$. This violates the condition of Corollary 12.6.3 required for K_5 to be planar, which proves

Lemma 12.6.4. *K_5 is not planar.*

Another consequence is

Lemma 12.6.5. *Every planar graph has a vertex of degree at most five.*

Proof. If every vertex had degree at least 6, then the sum of the vertex degrees is at least $6v$, but since the sum equals $2e$, we have $e \geq 3v$ contradicting the fact that $e \leq 3v - 6 < 3v$ by Corollary 12.6.3. ∎

12.7   Planar Subgraphs

If you draw a graph in the plane by repeatedly adding edges that don't cross, you clearly could add the edges in any other order and still wind up with the same drawing. This is so basic that we might presume that our recursively defined planar embeddings have this property. But that wouldn't be fair: we really need to prove it. After all, the recursive definition of planar embedding was pretty technical —maybe we got it a little bit wrong, with the result that our embeddings don't have this basic draw-in-any-order property.

Now any ordering of edges can be obtained just by repeatedly switching the order of successive edges, and if you think about the recursive definition of embedding for a minute, you should realize that you can switch *any* pair of successive edges if you can just switch the last two. So it all comes down to the following lemma.

Lemma 12.7.1. *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible by two successive applications of constructor operations to add edges e and then f to obtain a planar embedding, $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by adding f and then e with two successive applications of constructor operations.*

We'll leave the proof of Lemma 12.7.1 to Problem 12.6.

Corollary 12.7.2. *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible to add a sequence of edges $e_0, e_1, \ldots, e_n$ by successive applications of constructor operations to obtain a planar embedding, $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by applications of constructor operations that successively add any permutation[2] of the edges $e_0, e_1, \ldots, e_n$.*

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[2]If $\pi : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ is a bijection, then the sequence $e_{\pi(0)}, e_{\pi(1)}, \ldots, e_{\pi(n)}$ is called a *permutation* of the sequence $e_0, e_1, \ldots, e_n$.

Corollary 12.7.3. *Deleting an edge from a planar graph leaves a planar graph.*

Proof. By Corollary 12.7.2, we may assume the deleted edge was the last one added in constructing an embedding of the graph. So the embedding to which this last edge was added must be an embedding of the graph without that edge. ∎

Since we can delete a vertex by deleting all its incident edges, Corollary 12.7.3 immediately implies

Corollary 12.7.4. *Deleting a vertex from a planar graph, along with all its incident edges of course, leaves another planar graph.*

A *subgraph* of a graph, G, is any graph whose set of vertices is a subset of the vertices of G and whose set of edges is a subset of the set of edges of G. So we can summarize Corollaries 12.7.3 and 12.7.4 and their consequences in a Theorem.

Theorem 12.7.5. *Any subgraph of a planar graph is planar.*

12.8   Planar 5-Colorability

We need to know one more property of planar graphs in order to prove that planar graphs are 5-colorable.

Lemma 12.8.1. *Merging two adjacent vertices of a planar graph leaves another planar graph.*

Here merging two adjacent vertices, n_1 and n_2 of a graph means deleting the two vertices and then replacing them by a new "merged" vertex, m, adjacent to all the vertices that were adjacent to either of n_1 or n_2, as illustrated in Figure 12.7.

Lemma 12.8.1 can be proved by structural induction, but the proof is kind of boring, and we hope you'll be relieved that we're going to omit it. (If you insist, we can add it to the next problem set.)

Now we've got all the simple facts we need to prove 5-colorability.

Theorem 12.8.2. *Every planar graph is five-colorable.*

Proof. The proof will be by strong induction on the number, v, of vertices, with induction hypothesis:

Every planar graph with v vertices is five-colorable.

Base cases ($v \leq 5$): immediate.

Inductive case: Suppose G is a planar graph with $v + 1$ vertices. We will describe a five-coloring of G.

First, choose a vertex, g, of G with degree at most 5; Lemma 12.6.5 guarantees there will be such a vertex.

Case 1 ($\deg(g) < 5$): Deleting g from G leaves a graph, H, that is planar by Lemma 12.7.4, and, since H has v vertices, it is five-colorable by induction hypothesis. Now define a five coloring of G as follows: use the five-coloring of H for all
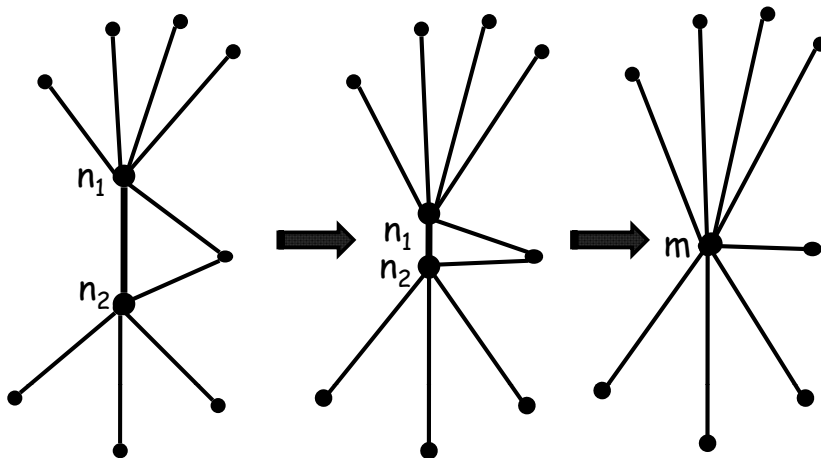
Figure 12.7: Merging adjacent vertices n_1 and n_2 into new vertex, m.

the vertices besides g, and assign one of the five colors to g that is not the same as the color assigned to any of its neighbors. Since there are fewer than 5 neighbors, there will always be such a color available for g.

Case 2 ($\deg(g) = 5$): If the five neighbors of g in G were all adjacent to each other, then these five vertices would form a nonplanar subgraph isomorphic to K_5, contradicting Theorem 12.7.5. So there must be two neighbors, n_1 and n_2, of g that are not adjacent. Now merge n_1 and g into a new vertex, m, as in Figure 12.7. In this new graph, n_2 is adjacent to m, and the graph is planar by Lemma 12.8.1. So we can then merge m and n_2 into a another new vertex, m', resulting in a new graph, G', which by Lemma 12.8.1 is also planar. Now G' has $v - 1$ vertices and so is five-colorable by the induction hypothesis.

Now define a five coloring of G as follows: use the five-coloring of G' for all the vertices besides g, n_1 and n_2. Next assign the color of m' in G' to be the color of the neighbors n_1 and n_2. Since n_1 and n_2 are not adjacent in G, this defines a proper five-coloring of G except for vertex g. But since these two neighbors of g have the same color, the neighbors of g have been colored using fewer than five colors altogether. So complete the five-coloring of G by assigning one of the five colors to g that is not the same as any of the colors assigned to its neighbors. ∎

A graph obtained from a graph, G, be repeatedly deleting vertices, deleting edges, and merging adjacent vertices is called a *minor* of G. Since K_5 and $K_{3,3}$ are not planar, Lemmas 12.7.3, 12.7.4, and 12.8.1 immediately imply:

Corollary 12.8.3. *A graph which has K_5 or $K_{3,3}$ as a minor is not planar.*

We don't have time to prove it, but the converse of Corollary 12.8.3 is also true. This gives the following famous, very elegant, and purely discrete characterization of planar graphs:

Theorem 12.8.4 (Kuratowksi). *A graph is not planar iff it has K_5 or $K_{3,3}$ as a minor.*

## 12.9 Classifying Polyhedra

The Pythagoreans had two great mathematical secrets, the irrationality of $\sqrt{2}$ and a geometric construct that we're about to rediscover!

A *polyhedron* is a convex, three-dimensional region bounded by a finite number of polygonal faces. If the faces are identical regular polygons and an equal number of polygons meet at each corner, then the polyhedron is *regular*. Three examples of regular polyhedra are shown below: the tetrahedron, the cube, and the octahedron.
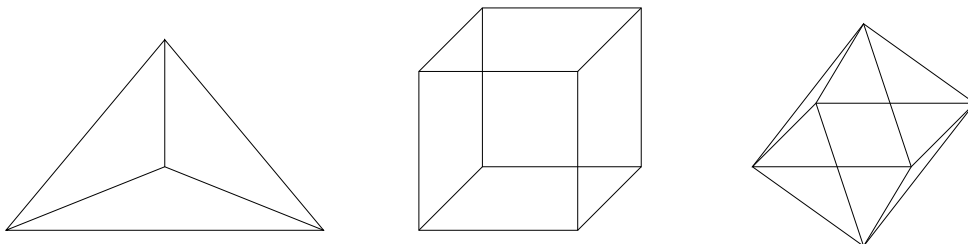
We can determine how many more regular polyhedra there are by thinking about planarity. Suppose we took *any* polyhedron and placed a sphere inside it. Then we could project the polyhedron face boundaries onto the sphere, which would give an image that was a planar graph embedded on the sphere, with the images of the corners of the polyhedron corresponding to vertices of the graph. But we've already observed that embeddings on a sphere are the same as embeddings on the plane, so Euler's formula for planar graphs can help guide our search for regular polyhedra.

For example, planar embeddings of the three polyhedra above look like this:
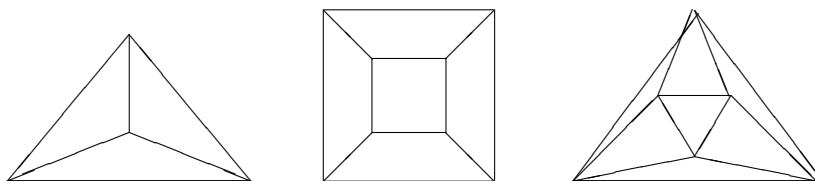
Let m be the number of faces that meet at each corner of a polyhedron, and let n be the number of sides on each face. In the corresponding planar graph, there are m edges incident to each of the v vertices. Since each edge is incident to two vertices, we know:

$$mv = 2e$$

Also, each face is bounded by n edges. Since each edge is on the boundary of two faces, we have:

$$nf = 2e$$

Solving for v and f in these equations and then substituting into Euler's formula gives:

$$\frac{2e}{m} - e + \frac{2e}{n} = 2$$

which simplifies to

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2} \tag{12.2}$$

This last equation (12.2) places strong restrictions on the structure of a polyhedron. Every nondegenerate polygon has at least 3 sides, so $n \geq 3$. And at least 3 polygons must meet to form a corner, so $m \geq 3$. On the other hand, if either n or m were 6 or more, then the left side of the equation could be at most $1/3 + 1/6 = 1/2$, which

is less than the right side. Checking the finitely-many cases that remain turns up only five solutions. For each valid combination of n and m, we can compute the associated number of vertices v, edges e, and faces f. And polyhedra with these properties do actually exist:

| n	m	v	e	f	polyhedron
3	3	4	6	4	tetrahedron
4	3	8	12	6	cube
3	4	6	12	8	octahedron
3	5	12	30	20	icosahedron
5	3	20	30	12	dodecahedron

The last polyhedron in this list, the dodecahedron, was the other great mathematical secret of the Pythagorean sect. These five, then, are the only possible regular polyhedra.

So if you want to put more than 20 geocentric satellites in orbit so that they *uniformly* blanket the globe —tough luck!

12.9.1 Problems

Exam Problems

Problem 12.1.
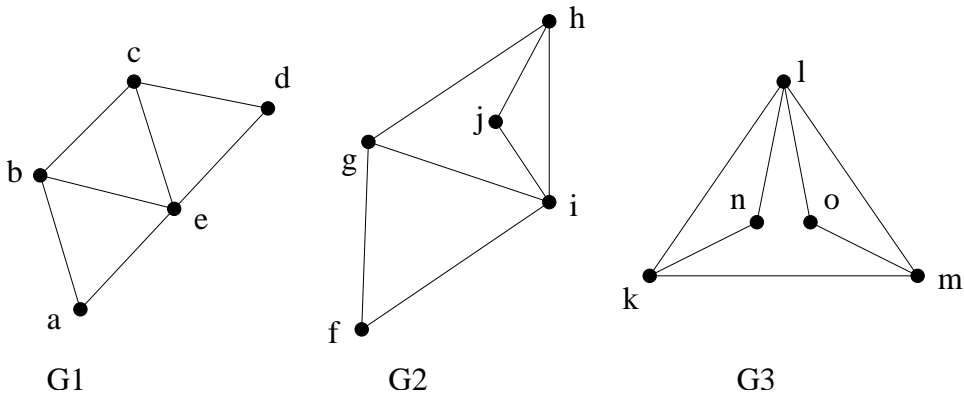
(a) Describe an isomorphism between graphs G_1 and G_2, and another isomorphism between G_2 and G_3.

(b) Why does part (a) imply that there is an isomorphism between graphs G_1 and G_3?

Let G and H be planar graphs. An embedding E_G of G is isomorphic to an embedding E_H of H iff there is an isomorphism from G to H that also maps each face of E_G to a face of E_H.
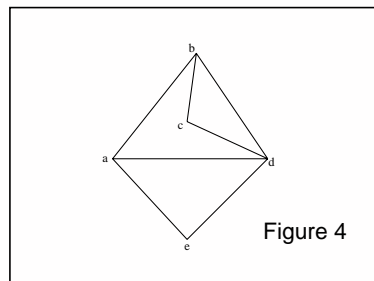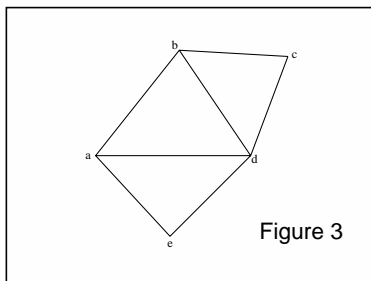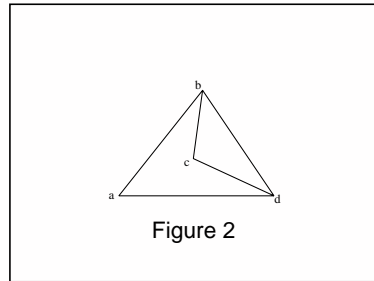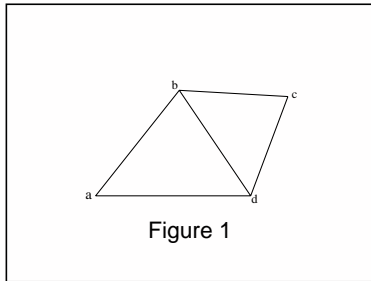
(c) One of the embeddings pictured above is not isomorphic to either of the others. Which one? Briefly explain why.

(d) Explain why all embeddings of two isomorphic planar graphs must have the same number of faces.

Class Problems

Problem 12.2.
Figures 1–4 show different pictures of planar graphs.

Figure 1

Figure 2

Figure 3

Figure 4

1

(a) For each picture, describe its discrete faces (simple cycles that define the region borders).

(b) Which of the pictured graphs are isomorphic? Which pictures represent the same *planar embedding*? – that is, they have the same discrete faces.

(c) Describe a way to construct the embedding in Figure 4 according to the recursive Definition 12.3.1 of planar embedding. For each application of a constructor rule, be sure to indicate the faces (cycles) to which the rule was applied and the cycles which result from the application.

Problem 12.3. (a) Show that if a connected planar graph with more than two vertices is bipartite, then

$$e \leq 2v - 4. \tag{12.3}$$

Hint: Similar to the proof of Corollary 12.6.3 that for planar graphs $e \leq 3v - 6$.

(b) Conclude that that $K_{3,3}$ is not planar. ($K_{3,3}$ is the graph with six vertices and an edge from each of the first three vertices to each of the last three.)

Problem 12.4.
Prove the following assertions by structural induction on the definition of planar embedding.

(a) In a planar embedding of a graph, each edge is traversed a total of two times by the faces of the embedding.

(b) In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

Homework Problems

Problem 12.5.
A simple graph is *triangle-free* when it has no simple cycle of length three.

(a) Prove for any connected triangle-free planar graph with $v > 2$ vertices and e edges, $e \leq 2v - 4$.

Hint: Similar to the proof that $e \leq 3v - 6$. Use Problem 12.4.

(b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

(c) Prove by induction on the number of vertices that any connected triangle-free planar graph is 4-colorable.

Hint: use part (b).

Problem 12.6. (a) Prove Lemma 12.7.1. *Hint:* There are four cases to analyze, depending on which two constructor operations are applied to add e and then f. Structural induction is not needed.

(b) Prove Corollary 12.7.2.

Hint: By induction on the number of switches of adjacent elements needed to convert the sequence 0,1,…,n into a permutation $\pi(0), \pi(1), \ldots, \pi(n)$.

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