Chapter 9

State Machines

9.1 State machines

State machines are an abstract model of step-by-step processes, and accordingly, they come up in many areas of computer science. You may already have seen them in a digital logic course, a compiler course, or a probability course.

9.1.1 Basic definitions

A state machine is really nothing more than a binary relation on a set, except that the elements of the set are called "states," the relation is called the *transition relation*, and a pair (p, q) in the graph of the transition relation is called a *transition*. The transition from state p to state q will be written $p \longrightarrow q$. The transition relation is also called the *state graph* of the machine. A state machine also comes equipped with a designated *start state*.

State machines used in digital logic and compilers usually have only a finite number of states, but machines that model continuing computations typically have an infinite number of states. In many applications, the states, and/or the transitions have labels indicating input or output values, costs, capacities, or probabilities, but for our purposes, unlabelled states and transitions are all we need.[1]

Example 9.1.1. A bounded counter, which counts from 0 to 99 and overflows at 100. The transitions are pictured in Figure 9.1, with start state zero. This machine isn't much use once it overflows, since it has no way to get out of its overflow state.

Example 9.1.2. An unbounded counter is similar, but has an infinite state set. This is harder to draw :-)

Example 9.1.3. In the movie *Die Hard 3: With a Vengeance*, the characters played by Samuel L. Jackson and Bruce Willis have to disarm a bomb planted by the diabolical Simon Gruber:

[1] We do name states, as in Figure 9.1, so we can talk about them, but the names aren't part of the state machine.

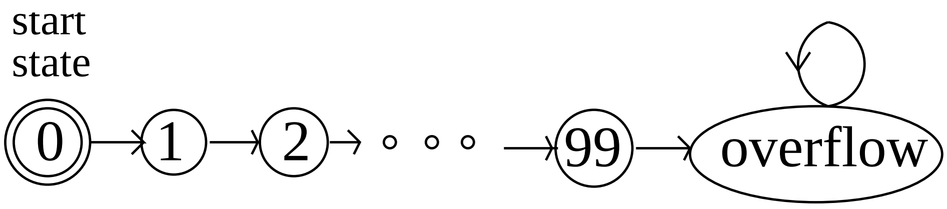


Figure 9.1: *State transitions for the 99-bounded counter.*

Simon: On the fountain, there should be 2 jugs, do you see them? A 5-gallon and a 3-gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

Bruce: Wait, wait a second. I don't get it. Do you get it?

Samuel: No.

Bruce: Get the jugs. Obviously, we can't fill the 3-gallon jug with 4 gallons of water.

Samuel: Obviously.

Bruce: All right. I know, here we go. We fill the 3-gallon jug exactly to the top, right?

Samuel: Uh-huh.

Bruce: Okay, now we pour this 3 gallons into the 5-gallon jug, giving us exactly 3 gallons in the 5-gallon jug, right?

Samuel: Right, then what?

Bruce: All right. We take the 3-gallon jug and fill it a third of the way...

Samuel: No! He said, "Be precise." Exactly 4 gallons.

Bruce: Sh - -. Every cop within 50 miles is running his a - - off and I'm out here playing kids games in the park.

Samuel: Hey, you want to focus on the problem at hand?

Fortunately, they find a solution in the nick of time. We'll let the reader work out how.

The *Die Hard* series is getting tired, so we propose a final *Die Hard Once and For All*. Here Simon's brother returns to avenge him, and he poses the same challenge, but with the 5 gallon jug replaced by a 9 gallon one.

We can model jug-filling scenarios with a state machine. In the scenario with a 3 and a 5 gallon water jug, the states will be pairs, (b, l) of real numbers such that

$0 \le b \le 5, 0 \le l \le 3$. We let b and l be arbitrary real numbers. (We can prove that the values of b and l will only be nonnegative integers, but we won't assume this.) The start state is $(0, 0)$, since both jugs start empty.

Since the amount of water in the jug must be known exactly, we will only consider moves in which a jug gets completely filled or completely emptied. There are several kinds of transitions:

1. Fill the little jug: $(b, l) \longrightarrow (b, 3)$ for $l < 3$.
2. Fill the big jug: $(b, l) \longrightarrow (5, l)$ for $b < 5$.
3. Empty the little jug: $(b, l) \longrightarrow (b, 0)$ for $l > 0$.
4. Empty the big jug: $(b, l) \longrightarrow (0, l)$ for $b > 0$.
5. Pour from the little jug into the big jug: for $l > 0$,
$$(b, l) \longrightarrow \begin{cases} (b + l, 0) & \text{if } b + l \le 5, \\ (5, l - (5 - b)) & \text{otherwise.} \end{cases}$$
6. Pour from big jug into little jug: for $b > 0$,
$$(b, l) \longrightarrow \begin{cases} (0, b + l) & \text{if } b + l \le 3, \\ (b - (3 - l), 3) & \text{otherwise.} \end{cases}$$

Note that in contrast to the 99-counter state machine, there is more than one possible transition out of states in the Die Hard machine. Machines like the 99-counter with at most one transition out of each state are called *deterministic*. The Die Hard machine is *nondeterministic* because some states have transitions to several different states.

Quick exercise: Which states of the Die Hard 3 machine have direct transitions to exactly two states?

9.1.2 Reachability and Preserved Invariants

The Die Hard 3 machine models every possible way of pouring water among the jugs according to the rules. Die Hard properties that we want to verify can now be expressed and proved using the state machine model. For example, Bruce's character will disarm the bomb if he can get to some state of the form $(4, l)$.

A (possibly infinite) path through the state graph beginning at the start state corresponds to a possible system behavior; such a path is called an *execution* of the state machine. A state is called *reachable* if it appears in some execution. The bomb in Die Hard 3 gets disarmed successfully because the state (4,3) is reachable.

A useful approach in analyzing state machine is to identify properties of states that are preserved by transitions.

Definition 9.1.4. A *preserved invariant* of a state machine is a predicate, P, on states, such that whenever $P(q)$ is true of a state, q, and $q \longrightarrow r$ for some state, r, then $P(r)$ holds.

The Invariant Principle

If a preserved invariant of a state machine is true for the start state, then it is true for all reachable states.

The Invariant Principle is nothing more than the Induction Principle reformulated in a convenient form for state machines. Showing that a predicate is true in the start state is the base case of the induction, and showing that a predicate is a preserved invariant is the inductive step.[2]

Die Hard Once and For All

Now back to Die Hard Once and For All. This time there is a 9 gallon jug instead of the 5 gallon jug. We can model this with a state machine whose states and transitions are specified the same way as for the Die Hard 3 machine, with all occurrences of "5" replaced by "9."

Now reaching any state of the form $(4, l)$ is impossible. We prove this using the Invariant Principle. Namely, we define the preserved invariant predicate, $P(b, l)$, to be that b and l are nonnegative integer multiples of 3. So P obviously holds for the state state $(0, 0)$.

To prove that P is a preserved invariant, we assume $P(b, l)$ holds for some state (b, l) and show that if $(b, l) \longrightarrow (b', l')$, then $P(b', l')$. The proof divides into cases, according to which transition rule is used. For example, suppose the transition followed from the "fill the little jug" rule. This means $(b, l) \longrightarrow (b, 3)$. But $P(b, l)$ implies that b is an integer multiple of 3, and of course 3 is an integer multiple of 3, so P still holds for the new state $(b, 3)$. Another example is when the transition rule used is "pour from big jug into little jug" for the subcase that $b + l > 3$. Then state is $(b, l) \longrightarrow (b - (3 - l), 3)$. But since b and l are integer multiples of 3, so is $b - (3 - l)$. So in this case too, P holds after the transition.

We won't bother to crank out the remaining cases, which can all be checked just as easily. Now by the Invariant Principle, we conclude that every reachable

[2]Preserved invariants are commonly just called "invariants" in the literature on program correctness, but we decided to throw in the extra adjective to avoid confusion with other definitions. For example, another subject at MIT uses "invariant" to mean "predicate true of all reachable states." Let's call this definition "invariant-2." Now invariant-2 seems like a reasonable definition, since unreachable states by definition don't matter, and all we want to show is that a desired property is invariant-2. But this confuses the *objective* of demonstrating that a property is invariant-2 with the *method* for showing that it is. After all, if we already knew that a property was invariant-2, we'd have no need for an Invariant Principle to demonstrate it.

state satisifies P. But since no state of the form $(4, l)$ satisifies P, we have proved rigorously that Bruce dies once and for all!

By the way, notice that the state (1,0), which satisfies $\text{NOT}(P)$, has a transition to (0,0), which satisfies P. So it's wrong to assume that the complement of a preserved invariant is also a preserved invariant.

A Robot on a Grid

There is a robot. It walks around on a grid, and at every step it moves diagonally in a way that changes its position by one unit up or down *and* one unit left or right. The robot starts at position $(0, 0)$. Can the robot reach position $(1, 0)$?

To get some intuition, we can simulate some robot moves. For example, starting at (0,0) the robot could move northeast to (1,1), then southeast to (2,0), then southwest to (1,-1), then southwest again to (0,-2).

Let's model the problem as a state machine and then find a suitable invariant. A state will be a pair of integers corresponding to the coordinates of the robot's position. State (i, j) has transitions to four different states: $(i \pm 1, j \pm 1)$.

The problem is now to choose an appropriate preserved invariant, P, that is true for the start state $(0, 0)$ and false for $(1, 0)$. The Invariant Theorem then will imply that the robot can never reach $(1, 0)$. A direct attempt for a preserved invariant is the predicate $P(q)$ that $q \neq (1, 0)$.

Unfortunately, this is not going to work. Consider the state $(2, 1)$. Clearly $P(2, 1)$ holds because $(2, 1) \neq (1, 0)$. And of course $P(1, 0)$ does not hold. But $(2, 1) \longrightarrow (1, 0)$, so this choice of P will not yield a preserved invariant.

We need a stronger predicate. Looking at our example execution you might be able to guess a proper one, namely, that the sum of the coordinates is even! If we can prove that this is a preserved invariant, then we have proven that the robot never reaches $(1, 0)$ —because the sum $1 + 0$ of its coordinates is odd, while the sum $0 + 0$ of the coordinates of the start state is even.

Theorem 9.1.5. *The sum of the robot's coordinates is always even.*

Proof. The proof uses the Invariant Principle.

Let $P(i, j)$ be the predicate that $i + j$ is even.

First, we must show that the predicate holds for the start state $(0, 0)$. Clearly, $P(0, 0)$ is true because $0 + 0$ is even.

Next, we must show that P is a preserved invariant. That is, we must show that for each transition $(i, j) \longrightarrow (i', j')$, if $i + j$ is even, then $i' + j'$ is even. But $i' = i \pm 1$ and $j' = j \pm 1$ by definition of the transitions. Therefore, $i' + j'$ is equal to $i + j$ or $i + j \pm 2$, all of which are even. ■

Corollary 9.1.6. *The robot cannot reach* $(1, 0)$.

Robert W. Floyd

The Invariant Principle was formulated by Robert Floyd at Carnegie Tech[a] in 1967. Floyd was already famous for work on formal grammars which transformed the field of programming language parsing; that was how he got to be a professor even though he never got a Ph.D. (He was admitted to a PhD program as a teenage prodigy, but flunked out and never went back.)

In that same year, Albert R. Meyer was appointed Assistant Professor in the Carnegie Tech Computer Science Department where he first met Floyd. Floyd and Meyer were the only theoreticians in the department, and they were both delighted to talk about their shared interests. After just a few conversations, Floyd's new junior colleague decided that Floyd was the smartest person he had ever met.

Naturally, one of the first things Floyd wanted to tell Meyer about was his new, as yet unpublished, Invariant Principle. Floyd explained the result to Meyer, and Meyer wondered (privately) how someone as brilliant as Floyd could be excited by such a trivial observation. Floyd had to show Meyer a bunch of examples before Meyer understood Floyd's excitement —not at the truth of the utterly obvious Invariant Principle, but rather at the insight that such a simple theorem could be so widely and easily applied in verifying programs.

Floyd left for Stanford the following year. He won the Turing award —the "Nobel prize" of computer science— in the late 1970's, in recognition both of his work on grammars and on the foundations of program verification. He remained at Stanford from 1968 until his death in September, 2001. You can learn more about Floyd's life and work by reading the eulogy written by his closest colleague, Don Knuth.

[a]The following year, Carnegie Tech was renamed Carnegie-Mellon Univ.

9.1.3 Sequential algorithm examples

Proving Correctness

Robert Floyd, who pioneered modern approaches to program verification, distinguished two aspects of state machine or process correctness:

1. The property that the final results, if any, of the process satisfy system requirements. This is called *partial correctness*.

   You might suppose that if a result was only partially correct, then it might also be partially incorrect, but that's not what he meant. The word "partial" comes from viewing a process that might not terminate as computing a *partial function*. So partial correctness means that when there is a result, it is correct, but the process might not always produce a result, perhaps because it gets stuck in a loop.

2. The property that the process always finishes, or is guaranteed to produce some legitimate final output. This is called *termination*.

Partial correctness can commonly be proved using the Invariant Principle. Termination can commonly be proved using the Well Ordering Principle. We'll illustrate Floyd's ideas by verifying the Euclidean Greatest Common Divisor (GCD) Algorithm.

The Euclidean Algorithm

The *Euclidean algorithm* is a three-thousand-year-old procedure to compute the greatest common divisor, $\gcd(a, b)$ of integers a and b. We can represent this algorithm as a state machine. A state will be a pair of integers (x, y) which we can think of as integer registers in a register program. The state transitions are defined by the rule

$$(x, y) \longrightarrow (y, \text{remainder}(x, y))$$

for $y \neq 0$. The algorithm terminates when no further transition is possible, namely when $y = 0$. The final answer is in x.

We want to prove:

1. Starting from the state with $x = a$ and $y = b > 0$, if we ever finish, then we have the right answer. That is, at termination, $x = \gcd(a, b)$. This is a *partial correctness* claim.

2. We do actually finish. This is a process *termination* claim.

Partial Correctness of GCD First let's prove that if GCD gives an answer, it is a correct answer. Specifically, let $d ::= \gcd(a, b)$. We want to prove that *if* the procedure finishes in a state (x, y), then $x = d$.

Proof. Define the state predicate

$$P(x, y) ::= \quad [\gcd(x, y) = d \text{ and } (x > 0 \text{ or } y > 0)].$$

P holds for the start state (a, b), by definition of d and the requirement that b is positive. Also, the preserved invariance of P follows immediately from

Lemma 9.1.7. *For all $m, n \in \mathbb{N}$ such that $n \neq 0$,*

$$\gcd(m, n) = \gcd(n, \text{remainder}(m, n)). \tag{9.1}$$

Lemma 9.1.7 is easy to prove: let q be the quotient and r be the remainder of m divided by n. Then $m = qn + r$ by definition. So any factor of both r and n will be a factor of m, and similarly any factor of both m and n will be a factor of r. So r, n and m, n have the same common factors and therefore the same gcd. Now by the Invariant Principle, P holds for all reachable states.

Since the only rule for termination is that $y = 0$, it follows that if (x, y) is a terminal state, then $y = 0$. If this terminal state is reachable, then the preserved invariant holds for (x, y). This implies that $\gcd(x, 0) = d$ and that $x > 0$. We conclude that $x = \gcd(x, 0) = d$. ■

Termination of GCD Now we turn to the second property, that the procedure must terminate. To prove this, notice that y gets strictly smaller after any one transition. That's because the value of y after the transition is the remainder of x divided by y, and this remainder is smaller than y by definition. But the value of y is always a nonnegative integer, so by the Well Ordering Principle, it reaches a minimum value among all its values at reachable states. But there can't be a transition from a state where y has its minimum value, because the transition would decrease y still further. So the reachable state where y has its minimum value is a state at which no further step is possible, that is, at which the procedure terminates.

Note that this argument does not prove that the minimum value of y is zero, only that the minimum value occurs at termination. But we already noted that the only rule for termination is that $y = 0$, so it follows that the minimum value of y must indeed be zero.

The Extended Euclidean Algorithm

An important fact about the $\gcd(a, b)$ is that it equals an integer linear combination of a and b, that is,

$$\gcd(a, b) = sa + tb \tag{9.2}$$

for some $s, t \in \mathbb{Z}$. We'll see some nice proofs of (9.2) later when we study Number Theory, but now we'll look at an extension of the Euclidean Algorithm that efficiently, if obscurely, produces the desired s and t. It is presented here simply as another example of application of the Invariant Method (plus, we'll need a procedure like this when we take up number theory based cryptography in a couple of weeks).

Don't worry if you find this Extended Euclidean Algorithm hard to follow, and you can't imagine where it came from. In fact, that's good, because this will illustrate an important point: given the right preserved invariant, you can verify programs you don't understand.

In particular, given nonnegative integers x and y, with $y > 0$, we claim the following procedure[3] halts with registers S and T containing integers s and t satisfying (9.2).

Inputs: $a, b \in \mathbb{N}, b > 0$.

Registers: X, Y, S, T, U, V, Q.

Extended Euclidean Algorithm:

```
X := a; Y := b; S := 0; T := 1; U := 1; V := 0;
loop:
```

[3]This procedure is adapted from Aho, Hopcroft, and Ullman's text on algorithms.

```
if Y divides X, then halt
else
  Q := quotient(X,Y);
         ;;the following assignments in braces are SIMULTANEOUS
 {X := Y,
  Y := remainder(X,Y);
  U := S,
  V := T,
  S := U - Q * S,
  T := V - Q * T};
goto loop;
```

Note that X, Y behave exactly as in the Euclidean GCD algorithm in Section 9.1.3, except that this extended procedure stops one step sooner, ensuring that $\gcd(x, y)$ is in Y at the end. So for all inputs x, y, this procedure terminates for the same reason as the Euclidean algorithm: the contents, y, of register Y is a nonnegative integer-valued variable that strictly decreases each time around the loop.

The following properties are preserved invariants that imply partial correctness:

$$\gcd(X, Y) = \gcd(a, b), \tag{9.3}$$
$$Sa + Tb = Y, \text{ and} \tag{9.4}$$
$$Ua + Vb = X. \tag{9.5}$$

To verify that these are preserved invariants, note that (9.3) is the same one we observed for the Euclidean algorithm. To check the other two properties, let x, y, s, t, u, v be the contents of registers X, Y, S, T, U, V at the start of the loop and assume that all the properties hold for these values. We must prove that (9.4) and (9.5) hold (we already know (9.3) does) for the new contents x', y', s', t', u', v' of these registers at the next time the loop is started.

Now according to the procedure, $u' = s, v' = t, x' = y$, so (9.5) holds for u', v', x' because of (9.4) for s, t, y. Also,

$$s' = u - qs, \quad t' = v - qt, \quad y' = x - qy$$

where $q = \text{quotient}(x, y)$, so

$$s'a + t'b = (u - qs)a + (v - qt)b = ua + vb - q(sa + tb) = x - qy = y',$$

and therefore (9.4) holds for s', t', y'.

Also, it's easy to check that all three preserved invariants are true just before the first time around the loop. Namely, at the start:

$$X = a, Y = b, S = 0, T = 1 \qquad \text{so}$$
$$Sa + Tb = 0a + 1b = b = Y \qquad \text{confirming (9.4).}$$

Also,

$$\begin{aligned} U = 1, V = 0, &\qquad \text{so} \\ Ua + Vb = 1a + 0b = a = X &\qquad \text{confirming (9.5).} \end{aligned}$$

Now by the Invariant Principle, they are true at termination. But at termination, the contents, Y, of register `Y` divides the contents, X, of register `X`, so preserved invariants (9.3) and (9.4) imply

$$\gcd(a, b) = \gcd(X, Y) = Y = Sa + Tb.$$

So we have the gcd in register `Y` and the desired coefficients in `S`, `T`.

Now we don't claim that this verification offers much insight. In fact, if you're not wondering how somebody came up with this concise program and invariant, you:

- are blessed with an inspired intellect allowing you to see how this program and its invariant were devised,
- have lost interest in the topic, or
- haven't read this far.

If none of the above apply to you, we can offer some reassurance by repeating that you're not expected to understand this program.

We've already observed that a preserved invariant is really just an induction hypothesis. As with induction, finding the right hypothesis is usually the hard part. We repeat:

> **Given the right preserved invariant, it can be easy to verify a program even if you don't understand it.**

We expect that the Extended Euclidean Algorithm presented above illustrates this point.

9.1.4 Derived Variables

The preceding termination proofs involved finding a nonnegative integer-valued measure to assign to states. We might call this measure the "size" of the state. We then showed that the size of a state decreased with every state transition. By the Well Ordering Principle, the size can't decrease indefinitely, so when a minimum size state is reached, there can't be any transitions possible: the process has terminated.

More generally, the technique of assigning values to states —not necessarily nonnegative integers and not necessarily decreasing under transitions— is often useful in the analysis of algorithms. *Potential functions* play a similar role in physics. In the context of computational processes, such value assignments for states are called *derived variables*.

For example, for the Die Hard machines we could have introduced a derived variable, f : states $\to \mathbb{R}$, for the amount of water in both buckets, by setting $f((a,b)) ::= a+b$. Similarly, in the robot problem, the position of the robot along the x-axis would be given by the derived variable x-coord, where x-coord$((i,j)) ::= i$.

We can formulate our general termination method as follows:

Definition 9.1.8. Let $\prec$ be a strict partial order on a set, A. A derived variable f : states $\to A$ is *strictly decreasing* iff

$$q \longrightarrow q' \text{ implies } f(q') \prec f(q).$$

We confirmed termination of the GCD and Extended GCD procedures by finding derived variables, y and $\mathtt{Y}$, respectively, that were nonnegative integer-valued and strictly decreasing. We can summarize this approach to proving termination as follows:

Theorem 9.1.9. *If f is a strictly decreasing $\mathbb{N}$-valued derived variable of a state machine, then the length of any execution starting at state q is at most $f(q)$.*

Of course we could prove Theorem 9.1.9 by induction on the value of $f(q)$, but think about what it says: "If you start counting down at some nonnegative integer $f(q)$, then you can't count down more than $f(q)$ times." Put this way, it's obvious.

Weakly Decreasing Variables

In addition being strictly decreasing, it will be useful to have derived variables with some other, related properties.

Definition 9.1.10. Let $\preceq$ be a weak partial order on a set, A. A derived variable $f : Q \to A$ is *weakly decreasing* iff

$$q \longrightarrow q' \text{ implies } f(q') \preceq f(q).$$

Strictly increasing and *weakly increasing* derived variables are defined similarly.[4]

9.1.5 Problems

Homework Problems

Problem 9.1.

You are given two buckets, A and B, a water hose, a receptacle, and a drain. The buckets and receptacle are initially empty. The buckets are labeled with their respectively capacities, positive integers a and b. The receptacle can be used to store an unlimited amount of water, but has no measurement markings. Excess water can be dumped into the drain. Among the possible moves are:

[4]Weakly increasing variables are often also called *nondecreasing*. We will avoid this terminology to prevent confusion between nondecreasing variables and variables with the much weaker property of *not* being a decreasing variable.

1. fill a bucket from the hose,
2. pour from the receptacle to a bucket until the bucket is full or the receptacle is empty, whichever happens first,
3. empty a bucket to the drain,
4. empty a bucket to the receptacle,
5. pour from A to B until either A is empty or B is full, whichever happens first,
6. pour from B to A until either B is empty or A is full, whichever happens first.

(a) Model this scenario with a state machine. (What are the states? How does a state change in response to a move?)

(b) Prove that we can put $k \in \mathbb{N}$ gallons of water into the receptacle using the above operations if and only if $\gcd(a, b) \mid k$. *Hint:* Use the fact that if a, b are positive integers then there exist integers s, t such that $\gcd(a, b) = sa + tb$ (see Notes 9.1.3).

Problem 9.2.
Here is a *very, very fun* game. We start with two distinct, positive integers written on a blackboard. Call them a and b. You and I now take turns. (I'll let you decide who goes first.) On each player's turn, he or she must write a new positive integer on the board that is the difference of two numbers that are already there. If a player can not play, then he or she loses.

For example, suppose that 12 and 15 are on the board initially. Your first play must be 3, which is $15 - 12$. Then I might play 9, which is $12 - 3$. Then you might play 6, which is $15 - 9$. Then I can not play, so I lose.

(a) Show that every number on the board at the end of the game is a multiple of $\gcd(a, b)$.

(b) Show that every positive multiple of $\gcd(a, b)$ up to $\max(a, b)$ is on the board at the end of the game.

(c) Describe a strategy that lets you win this game every time.

Problem 9.3.
In the late 1960s, the military junta that ousted the government of the small republic of Nerdia completely outlawed built-in multiplication operations, and also forbade division by any number other than 3. Fortunately, a young dissident found a way to help the population multiply any two nonnegative integers without risking persecution by the junta. The procedure he taught people is:

```
procedure multiply($x, y$: nonnegative integers)
$r := x$;
$s := y$;
$a := 0$;
while $s \neq 0$ do
    if $3 \mid s$ then
        $r := r + r + r$;
        $s := s/3$;
    else if $3 \mid (s - 1)$ then
        $a := a + r$;
        $r := r + r + r$;
        $s := (s - 1)/3$;
    else
        $a := a + r + r$;
        $r := r + r + r$;
        $s := (s - 2)/3$;
return $a$;
```

We can model the algorithm as a state machine whose states are triples of non-negative integers (r, s, a). The initial state is $(x, y, 0)$. The transitions are given by the rule that for $s > 0$:

$$(r, s, a) \rightarrow \begin{cases} (3r, s/3, a) & \text{if } 3 \mid s \\ (3r, (s-1)/3, a+r) & \text{if } 3 \mid (s-1) \\ (3r, (s-2)/3, a+2r) & \text{otherwise.} \end{cases}$$

(a) List the sequence of steps that appears in the execution of the algorithm for inputs $x = 5$ and $y = 10$.

(b) Use the Invariant Method to prove that the algorithm is partially correct—that is, if $s = 0$, then $a = xy$.

(c) Prove that the algorithm terminates after at most $1 + \log_3 y$ executions of the body of the do statement.

Problem 9.4.
A robot named Wall-E wanders around a two-dimensional grid. He starts out at $(0, 0)$ and is allowed to take four different types of step:

1. $(+2, -1)$
2. $(+1, -2)$
3. $(+1, +1)$

4. $(-3, 0)$

Thus, for example, Wall-E might walk as follows. The types of his steps are listed above the arrows.

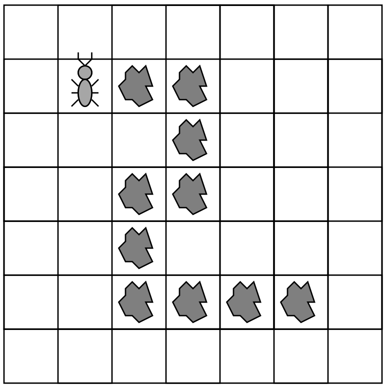
$$(0,0) \xrightarrow{1} (2,-1) \xrightarrow{3} (3,0) \xrightarrow{2} (4,-2) \xrightarrow{4} (1,-2) \to \dots$$

Wall-E's true love, the fashionable and high-powered robot, Eve, awaits at $(0, 2)$.

(a) Describe a state machine model of this problem.

(b) Will Wall-E ever find his true love? Either find a path from Wall-E to Eve or use the Invariant Principle to prove that no such path exists.

Problem 9.5.
A hungry ant is placed on an unbounded grid. Each square of the grid either contains a crumb or is empty. The squares containing crumbs form a path in which, except at the ends, every crumb is adjacent to exactly two other crumbs. The ant is placed at one end of the path and on a square containing a crumb. For example, the figure below shows a situation in which the ant faces North, and there is a trail of food leading approximately Southeast. The ant has already eaten the crumb upon which it was initially placed.



The ant can only smell food directly in front of it. The ant can only remember a small number of things, and what it remembers after any move only depends on what it remembered and smelled immediately before the move. Based on smell and memory, the ant may choose to move forward one square, or it may turn right or left. It eats a crumb when it lands on it.

The above scenario can be nicely modelled as a state machine in which each state is a pair consisting of the "ant's memory" and "everything else" —for example, information about where things are on the grid. Work out the details of such a model state machine; design the ant-memory part of the state machine so the ant will eat all the crumbs on *any* finite path at which it starts and then signal when it is done. Be sure to clearly describe the possible states, transitions, and inputs and outputs (if any) in your model. Briefly explain why your ant will eat all the crumbs.

Note that the last transition is a self-loop; the ant signals done for eternity. One could also add another end state so that the ant signals done only once.

Problem 9.6.
Suppose that you have a regular deck of cards arranged as follows, from top to bottom:

$$A\heartsuit\ 2\heartsuit \ldots K\heartsuit\ A\spadesuit\ 2\spadesuit \ldots K\spadesuit\ A\clubsuit\ 2\clubsuit \ldots K\clubsuit\ A\diamondsuit\ 2\diamondsuit \ldots K\diamondsuit$$

Only two operations on the deck are allowed: *inshuffling* and *outshuffling*. In both, you begin by cutting the deck exactly in half, taking the top half into your right hand and the bottom into your left. Then you shuffle the two halves together so that the cards are perfectly interlaced; that is, the shuffled deck consists of one card from the left, one from the right, one from the left, one from the right, etc. The top card in the shuffled deck comes from the right hand in an outshuffle and from the left hand in an inshuffle.

(a) Model this problem as a state machine.

(b) Use the Invariant Principle to prove that you can not make the entire first half of the deck black through a sequence of inshuffles and outshuffles.

Note: Discovering a suitable invariant can be difficult! The standard approach is to identify a bunch of reachable states and then look for a pattern, some feature that they all share.[5]

Problem 9.7.
The following procedure can be applied to any digraph, G:

1. Delete an edge that is traversed by a directed cycle.
2. Delete edge $u \to v$ if there is a directed path from vertex u to vertex v that does not traverse $u \to v$.
3. Add edge $u \to v$ if there is no directed path in either direction between vertex u and vertex v.

Repeat these operations until none of them are applicable.

This procedure can be modeled as a state machine. The start state is G, and the states are all possible digraphs with the same vertices as G.

[5] If this does not work, consider twitching and drooling until someone takes the problem away.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{1 \to 2, 2 \to 3, 3 \to 4, 3 \to 2, 1 \to 4\}$$

What are the possible final states reachable from G?

A *line graph* is a graph whose edges can all be traversed by a directed simple path. All the final graphs in part (a) are line graphs.

(b) Prove that if the procedure terminates with a digraph, H, then H is a line graph with the same vertices as G.

Hint: Show that if H is *not* a line graph, then some operation must be applicable.

(c) Prove that being a DAG is a preserved invariant of the procedure.

(d) Prove that if G is a DAG and the procedure terminates, then the path relation of the final line graph is a topological sort of G.

Hint: Verify that the predicate

$$P(u, v) ::= \text{there is a directed path from } u \text{ to } v$$

is a preserved invariant of the procedure, for any two vertices u, v of a DAG.

(e) Prove that if G is finite, then the procedure terminates.

Hint: Let s be the number of simple cycles, e be the number of edges, and p be the number of pairs of vertices with a directed path (in either direction) between them. Note that $p \leq n^2$ where n is the number of vertices of G. Find coefficients a, b, c such that $as + bp + e + c$ is a strictly decreasing, $\mathbb{N}$-valued variable.

Class Problems

Problem 9.8.
In this problem you will establish a basic property of a puzzle toy called the *Fifteen Puzzle* using the method of invariants. The Fifteen Puzzle consists of sliding square tiles numbered $1, \ldots, 15$ held in a 4×4 frame with one empty square. Any tile adjacent to the empty square can slide into it.

The standard initial position is

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

We would like to reach the target position (known in my youth as "the impossible" — ARM):

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	

A state machine model of the puzzle has states consisting of a 4×4 matrix with 16 entries consisting of the integers $1, \ldots, 15$ as well as one "empty" entry—like each of the two arrays above.

The state transitions correspond to exchanging the empty square and an adjacent numbered tile. For example, an empty at position $(2, 2)$ can exchange position with tile above it, namely, at position $(1, 2)$:

$$\begin{array}{|c|c|c|c|}\hline n_1 & n_2 & n_3 & n_4 \\ \hline n_5 & & n_6 & n_7 \\ \hline n_8 & n_9 & n_{10} & n_{11} \\ \hline n_{12} & n_{13} & n_{14} & n_{15} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline n_1 & & n_3 & n_4 \\ \hline n_5 & n_2 & n_6 & n_7 \\ \hline n_8 & n_9 & n_{10} & n_{11} \\ \hline n_{12} & n_{13} & n_{14} & n_{15} \\ \hline \end{array}$$

We will use the invariant method to prove that there is no way to reach the target state starting from the initial state.

We begin by noting that a state can also be represented as a pair consisting of two things:

1. a list of the numbers $1, \ldots, 15$ in the order in which they appear—reading rows left-to-right from the top row down, ignoring the empty square, and

2. the coordinates of the empty square—where the upper left square has coordinates $(1, 1)$, the lower right $(4, 4)$.

(a) Write out the "list" representation of the start state and the "impossible" state.

Let L be a list of the numbers $1, \ldots, 15$ in some order. A pair of integers is an *out-of-order pair* in L when the first element of the pair both comes *earlier* in the list and *is larger*, than the second element of the pair. For example, the list $1, 2, 4, 5, 3$ has two out-of-order pairs: (4,3) and (5,3). The increasing list $1, 2 \ldots n$ has no out-of-order pairs.

Let a state, S, be a pair $(L, (i, j))$ described above. We define the *parity* of S to be the mod 2 sum of the number, $p(L)$, of out-of-order pairs in L and the row-number of the empty square, that is the parity of S is $p(L) + i \pmod 2$.

(b) Verify that the parity of the start state and the target state are different.

(c) Show that the parity of a state is preserved under transitions. Conclude that "the impossible" is impossible to reach.

By the way, if two states have the same parity, then in fact there *is* a way to get from one to the other. If you like puzzles, you'll enjoy working this out on your own.

Problem 9.9.

The most straightforward way to compute the bth power of a number, a, is to multiply a by itself b times. This of course requires $b - 1$ multiplications. There is another way to do it using considerably fewer multiplications. This algorithm is called *fast exponentiation*:

Given inputs $a \in \mathbb{R}, b \in \mathbb{N}$, initialize registers x, y, z to $a, 1, b$ respectively, and repeat the following sequence of steps until termination:

- if $z = 0$ **return** y and terminate
- $r :=$ remainder$(z, 2)$
- $z := \text{quotient}(z, 2)$
- if $r = 1$, then $y := xy$
- $x := x^2$

We claim this algorithm always terminates and leaves $y = a^b$.

(a) Model this algorithm with a state machine, carefully defining the states and transitions.

(b) Verify that the predicate $P((x, y, z)) ::= [yx^z = a^b]$ is a preserved invariant.

(c) Prove that the algorithm is partially correct: if it halts, it does so with $y = a^b$.

(d) Prove that the algorithm terminates.

(e) In fact, prove that it requires at most $2 \lceil \log_2(b + 1) \rceil$ multiplications for the Fast Exponentiation algorithm to compute a^b for $b > 1$.

Problem 9.10.
A robot moves on the two-dimensional integer grid. It starts out at $(0, 0)$, and is allowed to move in any of these four ways:

1. (+2,-1) Right 2, down 1
2. (-2,+1) Left 2, up 1
3. (+1,+3)
4. (-1,-3)

Prove that this robot can never reach (1,1).

Problem 9.11.
The Massachusetts Turnpike Authority is concerned about the integrity of the new Zakim bridge. Their consulting architect has warned that the bridge may collapse if more than 1000 cars are on it at the same time. The Authority has also been warned by their traffic consultants that the rate of accidents from cars speeding across bridges has been increasing.

Both to lighten traffic and to discourage speeding, the Authority has decided to make the bridge *one-way* and to put tolls at *both* ends of the bridge (don't laugh, this is Massachusetts). So cars will pay tolls both on entering and exiting the bridge, but the tolls will be different. In particular, a car will pay \$3 to enter onto the bridge and will pay \$2 to exit. To be sure that there are never too many cars on the bridge, the Authority will let a car onto the bridge only if the difference between the amount of money currently at the entry toll booth minus the amount at the exit toll booth is strictly less than a certain threshold amount of \$$T_0$.

The consultants have decided to model this scenario with a state machine whose states are triples of natural numbers, (A, B, C), where

- A is an amount of money at the entry booth,
- B is an amount of money at the exit booth, and
- C is a number of cars on the bridge.

Any state with $C > 1000$ is called a *collapsed* state, which the Authority dearly hopes to avoid. There will be no transition out of a collapsed state.

Since the toll booth collectors may need to start off with some amount of money in order to make change, and there may also be some number of "official" cars already on the bridge when it is opened to the public, the consultants must be ready to analyze the system started at *any* uncollapsed state. So let A_0 be the initial number of dollars at the entrance toll booth, B_0 the initial number of dollars at the exit toll booth, and $C_0 \leq 1000$ the number of official cars on the bridge when it is opened. You should assume that even official cars pay tolls on exiting or entering the bridge after the bridge is opened.

(a) Give a mathematical model of the Authority's system for letting cars on and off the bridge by specifying a transition relation between states of the form (A, B, C) above.

(b) Characterize each of the following derived variables

$$A, B, A+B, A-B, 3C-A, 2A-3B, B+3C, 2A-3B-6C, 2A-2B-3C$$

as one of the following

constant	C
strictly increasing	SI
strictly decreasing	SD
weakly increasing but not constant	WI
weakly decreasing but not constant	WD
none of the above	N

and briefly explain your reasoning.

The Authority has asked their engineering consultants to determine T and to verify that this policy will keep the number of cars from exceeding 1000.

The consultants reason that if C_0 is the number of official cars on the bridge when it is opened, then an additional $1000 - C_0$ cars can be allowed on the bridge. So as long as $A - B$ has not increased by $3(1000 - C_0)$, there shouldn't more than 1000 cars on the bridge. So they recommend defining

$$T_0 ::= 3(1000 - C_0) + (A_0 - B_0), \tag{9.6}$$

where A_0 is the initial number of dollars at the entrance toll booth, B_0 is the initial number of dollars at the exit toll booth.

(c) Use the results of part (b) to define a simple predicate, P, on states of the transition system which is satisfied by the start state, that is $P(A_0, B_0, C_0)$ holds, is not satisfied by any collapsed state, and is a preserved invariant of the system. Explain why your P has these properties.

(d) A clever MIT intern working for the Turnpike Authority agrees that the Turnpike's bridge management policy will be *safe*: the bridge will not collapse. But she warns her boss that the policy will lead to *deadlock*— a situation where traffic can't move on the bridge even though the bridge has not collapsed.

Explain more precisely in terms of system transitions what the intern means, and briefly, but clearly, justify her claim.

Problem 9.12.
Start with 102 coins on a table, 98 showing heads and 4 showing tails. There are two ways to change the coins:

(i) flip over any ten coins, or

(ii) let n be the number of heads showing. Place $n+1$ additional coins, all showing tails, on the table.

For example, you might begin by flipping nine heads and one tail, yielding 90 heads and 12 tails, then add 91 tails, yielding 90 heads and 103 tails.

(a) Model this situation as a state machine, carefully defining the set of states, the start state, and the possible state transitions.

(b) Explain how to reach a state with exactly one tail showing.

(c) Define the following derived variables:

C	::=	the number of coins on the table,	H	::=	the number of heads,
T	::=	the number of tails,	C_2	::=	$\text{remainder}(C/2)$,
H_2	::=	$\text{remainder}(H/2)$,	T_2	::=	$\text{remainder}(T/2)$.

Which of these variables is

1. strictly increasing
2. weakly increasing
3. strictly decreasing
4. weakly decreasing
5. constant

(d) Prove that it is not possible to reach a state in which there is exactly one head showing.

Problem 9.13.
In some terms when 6.042 is not taught in a TEAL room, students sit in a square arrangement during recitations. An outbreak of beaver flu sometimes infects students in recitation; beaver flu is a rare variant of bird flu that lasts forever, with symptoms including a yearning for more quizzes and the thrill of late night problem set sessions.

Here is an example of a 6×6 recitation arrangement with the locations of infected students marked with an asterisk.

*				*	
	*				
		*	*		
		*			
			*		*

Outbreaks of infection spread rapidly step by step. A student is infected after a step if either

- the student was infected at the previous step (since beaver flu lasts forever), or
- the student was adjacent to *at least two* already-infected students at the previous step.

Here *adjacent* means the students' individual squares share an edge (front, back, left or right, but *not* diagonal). Thus, each student is adjacent to 2, 3 or 4 others.

In the example, the infection spreads as shown below.

*				*	
	*				
		*	*		
		*			
			*		*

$\Rightarrow$

*	*			*	
*	*	*			
	*	*	*		
		*			
		*	*		
		*	*	*	*

$\Rightarrow$

*	*	*		*	
*	*	*	*		
*	*	*	*		
	*	*	*		
		*	*	*	
		*	*	*	*

In this example, over the next few time-steps, all the students in class become infected.

Theorem. *If fewer than n students among those in an $n \times n$ arrangment are initially infected in a flu outbreak, then there will be at least one student who never gets infected in this outbreak, even if students attend all the lectures.*

Prove this theorem.

Hint: Think of the state of an outbreak as an $n \times n$ square above, with asterisks indicating infection. The rules for the spread of infection then define the transitions of a state machine. Try to derive a weakly decreasing state variable that leads to a proof of this theorem.

9.2 The Stable Marriage Problem

Okay, frequent public reference to derived variables may not help your mating prospects. But they can help with the analysis!

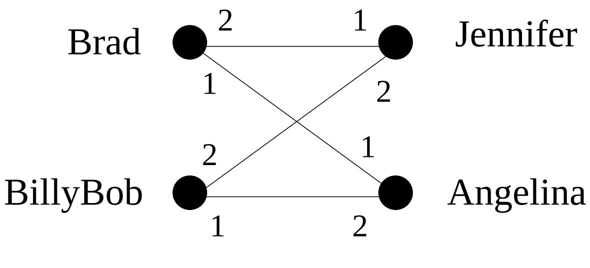
9.2.1 The Problem

Suppose there are a bunch of boys and an equal number of girls that we want to marry off. Each boy has his personal preferences about the girls —in fact, we assume he has a complete list of all the girls ranked according to his preferences, with no ties. Likewise, each girl has a ranked list of all of the boys.

The preferences don't have to be symmetric. That is, Jennifer might like Brad best, but Brad doesn't necessarily like Jennifer best. The goal is to marry off boys and girls: every boy must marry exactly one girl and vice-versa—no polygamy. In mathematical terms, we want the mapping from boys to their wives to be a bijection or *perfect matching*. We'll just call this a "matching," for short.

Here's the difficulty: suppose *every* boy likes Angelina best, and every girl likes Brad best, but Brad and Angelina are married to other people, say Jennifer and Billy Bob. Now *Brad and Angelina prefer each other to their spouses*, which puts their marriages at risk: pretty soon, they're likely to start spending late nights doing 6.042 homework together.

This situation is illustrated in the following diagram where the digits "1" and "2" near a boy shows which of the two girls he ranks first and which second, and similarly for the girls:



More generally, in any matching, a boy and girl who are not married to each other and who like each other better than their spouses, is called a *rogue couple*. In the situation above, Brad and Angelina would be a rogue couple.

Having a rogue couple is not a good thing, since it threatens the stability of the marriages. On the other hand, if there are no rogue couples, then for any boy and girl who are not married to each other, at least one likes their spouse better than the other, and so won't be tempted to start an affair.

Definition 9.2.1. A *stable matching* is a matching with no rogue couples.

The question is, given everybody's preferences, how do you find a stable set of marriages? In the example consisting solely of the four people above, we could let Brad and Angelina both have their first choices by marrying each other. Now neither Brad nor Angelina prefers anybody else to their spouse, so neither will be in a rogue couple. This leaves Jen not-so-happily married to Billy Bob, but neither Jen nor Billy Bob can entice somebody else to marry them.

It is something of a surprise that there always is a stable matching among a group of boys and girls, but there is, and we'll shortly explain why. The surprise springs in part from considering the apparently similar "buddy" matching problem. That is, if people can be paired off as buddies, regardless of gender, then a stable matching *may not* be possible. For example, Figure 9.2 shows a situation with a love triangle and a fourth person who is everyone's last choice. In this figure Mergatoid's preferences aren't shown because they don't even matter.

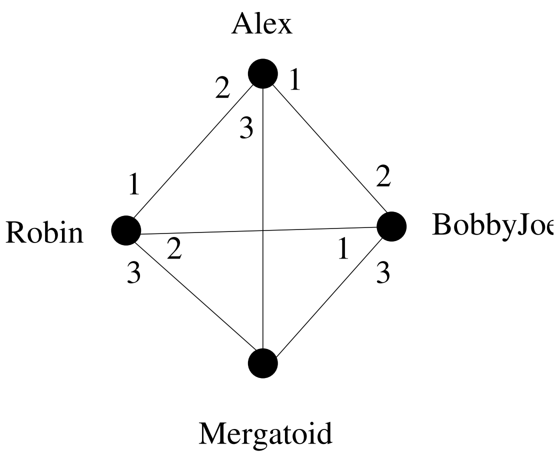


Figure 9.2: Some preferences with no stable buddy matching.

Let's see why there is no stable matching:

Lemma. *There is no stable buddy matching among the four people in Figure 9.2.*

Proof. We'll prove this by contradiction.

Assume, for the purposes of contradiction, that there is a stable matching. Then there are two members of the love triangle that are matched. Since preferences in the triangle are symmetric, we may assume in particular, that Robin and Alex are matched. Then the other pair must be Bobby-Joe matched with Mergatoid.

But then there is a rogue couple: Alex likes Bobby-Joe best, and Bobby-Joe prefers Alex to his buddy Mergatoid. That is, Alex and Bobby-Joe are a rogue couple, contradicting the assumed stability of the matching. ■

So getting a stable *buddy* matching may not only be hard, it may be impossible. But when boys are only allowed to marry girls, and vice versa, then it turns out that a stable matching is not hard to find.

9.2.2 The Mating Ritual

The procedure for finding a stable matching involves a *Mating Ritual* that takes place over several days. The following events happen each day:

Morning: Each girl stands on her balcony. Each boy stands under the balcony of his favorite among the girls on his list, and he serenades her. If a boy has no girls left on his list, he stays home and does his 6.042 homework.

Afternoon: Each girl who has one or more suitors serenading her, says to her favorite among them, "We might get engaged. Come back tomorrow." To the other suitors, she says, "No. I will never marry you! Take a hike!"

Evening: Any boy who is told by a girl to take a hike, crosses that girl off his list.

Termination condition: When every girl has at most one suitor, the ritual ends with each girl marrying her suitor, if she has one.

There are a number of facts about this Mating Ritual that we would like to prove:

- The Ritual has a last day.
- Everybody ends up married.
- The resulting marriages are stable.

9.2.3 A State Machine Model

Before we can prove anything, we should have clear mathematical definitions of what we're talking about. In this section we sketch how to define a rigorous state machine model of the Marriage Problem.

So let's begin by formally defining the problem.

Definition 9.2.2. A *Marriage Problem* consists of two disjoint sets of the same finite size, called the-Boys and the-Girls. The members of the-Boys are called *boys*, and members of the-Girls are called *girls*. For each boy, B, there is a strict total order, $<_B$, on the-Girls, and for each girl, G, there is a strict total order, $<_G$, on the-Boys. If $G_1 <_B G_2$ we say B *prefers* girl G_2 to girl G_1. Similarly, if $B_1 <_G B_2$ we say G *prefers* boy B_2 to boy B_1.

A *marriage assignment* or *perfect matching* is a bijection, w : the-Boys $\to$ the-Girls. If $B \in$ the-Boys, then $w(B)$ is called B's *wife* in the assignment, and if $G \in$ the-Girls, then $w^{-1}(G)$ is called G's *husband*. A *rogue couple* is a boy, B, and a girl, G, such that B prefers G to his wife, and G prefers B to her husband. An assignment is *stable* if it has no rogue couples. A *solution* to a marriage problem is a stable perfect matching.

To model the Mating Ritual with a state machine, we make a key observation: to determine what happens on any day of the Ritual, all we need to know is which girls are still on which boys' lists on the morning of that day. So we define a state to be some mathematical data structure providing this information. For example, we could define a state to be the "still-has-on-his-list" relation, R, between boys and girls, where $B \ R \ G$ means girl G is still on boy B's list.

We start the Mating Ritual with no girls crossed off. That is, the start state is the *complete bipartite* relation in which every boy is related to every girl.

According to the Mating Ritual, on any given morning, a boy will *serenade* the girl he most prefers among those he has not as yet crossed out. Mathematically, the girl he is serenading is just the maximum among the girls on B's list, ordered by $<_B$. (If the list is empty, he's not serenading anybody.) A girl's *favorite* is just the maximum, under her preference ordering, among the boys serenading her.

Continuing in this way, we could mathematically specify a precise Mating Ritual state machine, but we won't bother. The intended behavior of the Mating Ritual is clear enough that we don't gain much by giving a formal state machine, so we stick to a more memorable description in terms of boys, girls, and their preferences. The point is, though, that it's not hard to define everything using basic mathematical data structures like sets, functions, and relations, if need be.

9.2.4 There is a Marriage Day

It's easy to see why the Mating Ritual has a terminal day when people finally get married. Every day on which the ritual hasn't terminated, at least one boy crosses a girl off his list. (If the ritual hasn't terminated, there must be some girl serenaded by at least two boys, and at least one of them will have to cross her off his list). So starting with n boys and n girls, each of the n boys' lists initially has n girls on it, for a total of n^2 list entries. Since no girl ever gets added to a list, the total number of entries on the lists decreases every day that the Ritual continues, and so the Ritual can continue for at most n^2 days.

9.2.5 They All Live Happily Every After...

We still have to prove that the Mating Ritual leaves everyone in a stable marriage. To do this, we note one very useful fact about the Ritual: if a girl has a favorite boy suitor on some morning of the Ritual, then that favorite suitor will still be serenading her the next morning —because his list won't have changed. So she is sure to have today's favorite boy among her suitors tomorrow. That means she will be able to choose a favorite suitor tomorrow who is at least as desirable to her as today's favorite. So day by day, her favorite suitor can stay the same or get better, never worse. In others words, a girl's favorite is a weakly increasing variable with respect to her preference order on the boys.

Now we can verify the Mating Ritual using a simple invariant predicate, P, that captures what's going on:

> For every girl, G, and every boy, B, if G is crossed off B's list, then G has a suitor whom she prefers over B.

Why is P invariant? Well, we know that G's favorite tomorrow will be at least as desirable to her as her favorite today, and since her favorite today is more desirable than B, tomorrow's favorite will be too.

Notice that P also holds on the first day, since every girl is on every list. So by the Invariant Theorem, we know that P holds on every day that the Mating Ritual runs. Knowing the invariant holds when the Mating Ritual terminates will let us complete the proofs.

Theorem 9.2.3. *Everyone is married by the Mating Ritual.*

Proof. Suppose, for the sake of contradiction, that it is the last day of the Mating Ritual and some boy does not get married. Then he can't be serenading anybody, and so his list must be empty. So by invariant P, every girl has a favorite boy whom she prefers to that boy. In particular, every girl has a favorite boy whom she marries on the last day. So all the girls are married. What's more there is no bigamy: a boy only serenades one girl, so no two girls have the same favorite.

But there are the same number of girls as boys, so all the boys must be married too. ■

Theorem 9.2.4. *The Mating Ritual produces a stable matching.*

Proof. Let Brad be some boy and Jen be any girl that he is *not* married to on the last day of the Mating Ritual. We claim that Brad and Jen are not a rogue couple. Since Brad is an arbitrary boy, it follows that no boy is part of a rogue couple. Hence the marriages on the last day are stable.

To prove the claim, we consider two cases:

Case 1. Jen is not on Brad's list. Then by invariant P, we know that Jen prefers her husband to Brad. So she's not going to run off with Brad: the claim holds in this case.

Case 2. Otherwise, Jen is on Brad's list. But since Brad is not married to Jen, he must be choosing to serenade his wife instead of Jen, so he must prefer his wife. So he's not going to run off with Jen: the claim also holds in this case. ■

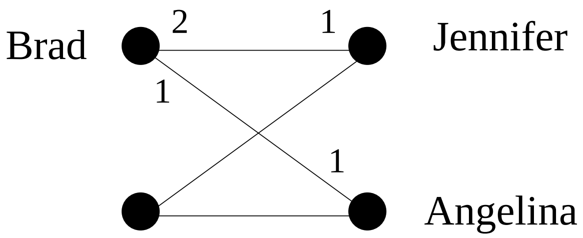
9.2.6 ...Especially the Boys

Who is favored by the Mating Ritual, the boys or the girls? The girls seem to have all the power: they stand on their balconies choosing the finest among their suitors and spurning the rest. What's more, we know their suitors can only change for the better as the Ritual progresses. Similarly, a boy keeps serenading the girl he most prefers among those on his list until he must cross her off, at which point he serenades the next most preferred girl on his list. So from the boy's point of view, the girl he is serenading can only change for the worse. Sounds like a good deal for the girls.

But it's not! The fact is that from the beginning, the boys are serenading their first choice girl, and the desirability of the girl being serenaded decreases only enough to give the boy his most desirable possible spouse. The mating algorithm actually does as well as possible for all the boys and does the worst possible job for the girls.

To explain all this we need some definitions. Let's begin by observing that while the mating algorithm produces one stable matching, there may be other stable matchings among the same set of boys and girls. For example, reversing the roles of boys and girls will often yield a different stable matching among them.

But some spouses might be out of the question in all possible stable matchings. For example, Brad is just not in the realm of possibility for Jennifer, since if you ever pair them, Brad and Angelina will form a rogue couple; here's a picture:



Definition 9.2.5. Given any marriage problem, one person is in another person's *realm of possible spouses* if there is a stable matching in which the two people are married. A person's *optimal spouse* is their most preferred person within their realm of possibility. A person's *pessimal spouse* is their least preferred person in their realm of possibility.

Everybody has an optimal and a pessimal spouse, since we know there is at least one stable matching, namely the one produced by the Mating Ritual. Now here is the shocking truth about the Mating Ritual:

Theorem 9.2.6. *The Mating Ritual marries every boy to his optimal spouse.*

Proof. Assume for the purpose of contradiction that some boy does not get his optimal girl. There must have been a day when he crossed off his optimal girl —otherwise he would still be serenading her or some even more desirable girl.

By the Well Ordering Principle, there must be a *first* day when a boy, call him "Keith," crosses off his optimal girl, Nicole.

According to the rules of the Ritual, Keith crosses off Nicole because Nicole has a favorite suitor, Tom, and

Nicole prefers Tom to Keith (*)

(remember, this is a proof by contradiction `:-)`).

Now since this is the first day an optimal girl gets crossed off, we know Tom hasn't crossed off his optimal girl. So

Tom ranks Nicole at least as high as his optimal girl. (**)

By the definition of an optimal girl, there must be some stable set of marriages in which Keith gets his optimal girl, Nicole. But then the preferences given in (*) and (**) imply that Nicole and Tom are a rogue couple within this supposedly stable set of marriages (think about it). This is a contradiction. ■

Theorem 9.2.7. *The Mating Ritual marries every girl to her pessimal spouse.*

Proof. Say Nicole and Keith marry each other as a result of the Mating Ritual. By the previous Theorem 9.2.6, Nicole is Keith's optimal spouse, and so in any stable set of marriages,

Keith rates Nicole at least as high as his spouse. (+)

Now suppose for the purpose of contradiction that there is another stable set of marriages where Nicole does worse than Keith. That is, Nicole is married to Tom, and

Nicole prefers Keith to Tom (++)

Then in this stable set of marriages where Nicole is married to Tom, (+) and (++) imply that Nicole and Keith are a rogue couple, contradicting stability. We conclude that Nicole cannot do worse than Keith. ■

9.2.7 Applications

Not surprisingly, a stable matching procedure is used by at least one large dating agency. But although "boy-girl-marriage" terminology is traditional and makes some of the definitions easier to remember (we hope without offending anyone), solutions to the Stable Marriage Problem are widely useful.

The Mating Ritual was first announced in a paper by D. Gale and L.S. Shapley in 1962, but ten years before the Gale-Shapley paper was appeared, and unknown by them, the Ritual was being used to assign residents to hospitals by the National Resident Matching Program (NRMP). The NRMP has, since the turn of the twentieth century, assigned each year's pool of medical school graduates to hospital residencies (formerly called "internships") with hospitals and graduates playing the roles of boys and girls. (In this case there may be multiple boys married to one girl, but there's an easy way to use the Ritual in this situation (see Problem 9.18). Before the Ritual was adopted, there were chronic disruptions and awkward countermeasures taken to preserve assignments of graduates to residencies. The Ritual resolved these problems so successfully, that it was used essentially without change at least through 1989.[6]

MIT Math Prof. Tom Leighton, who regularly teaches 6.042 and also founded the internet infrastructure company, Akamai, reports another application. Akamai uses a variation of the Gale-Shapley procedure to assign web traffic to servers. In the early days, Akamai used other combinatorial optimization algorithms that got to be too slow as the number of servers and traffic increased. Akamai switched to Gale-Shapley since it is fast and can be run in a distributed manner. In this case, the web traffic corresponds to the boys and the web servers to the girls. The servers have preferences based on latency and packet loss; the traffic has preferences based on the cost of bandwidth.

[6]Much more about the Stable Marriage Problem can be found in the very readable mathematical monograph by Dan Gusfield and Robert W. Irving, The Stable Marriage Problem: Structure and Algorithms, MIT Press, Cambridge, Massachusetts, 1989, 240 pp.

9.2.8 Problems

Practice Problems

Problem 9.14.
Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
Albert:	HP, Bellcore, AT&T, Draper
Rich:	AT&T, Bellcore, Draper, HP
Megumi:	HP, Draper, AT&T, Bellcore
Justin:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Justin, Albert, Megumi, Rich
Bellcore:	Megumi, Rich, Albert, Justin
HP:	Justin, Megumi, Albert, Rich
Draper:	Rich, Justin, Megumi, Albert

(a) Use the Mating Ritual to find *two* stable assignments of Students to Companies.

(b) Describe a simple procedure to determine whether any given stable marriage problem has a unique solution, that is, only one possible stable matching.

Problem 9.15.
Suppose that Harry is one of the boys and Alice is one of the girls in the *Mating Ritual*. Which of the properties below are preserved invariants? Why?

a. Alice is the only girl on Harry's list.

b. There is a girl who does not have any boys serenading her.

c. If Alice is not on Harry's list, then Alice has a suitor that she prefers to Harry.

d. Alice is crossed off Harry's list and Harry prefers Alice to anyone he is serenading.

e. If Alice is on Harry's list, then she prefers to Harry to any suitor she has.

Class Problems

Problem 9.16.
A preserved invariant of the Mating ritual is:

> For every girl, G, and every boy, B, if G is crossed off B's list, then G has a favorite suitor and she prefers him over B.

Use the invariant to prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Notes or slides.)

Problem 9.17.
Consider a stable marriage problem with 4 boys and 4 girls and the following partial information about their preferences:

B1:	G1	G2	–	–
B2:	G2	G1	–	–
B3:	–	–	G4	G3
B4:	–	–	G3	G4
G1:	B2	B1	–	–
G2:	B1	B2	–	–
G3:	–	–	B3	B4
G4:	–	–	B4	B3

(a) Verify that

$$(B1, G1), (B2, G2), (B3, G3), (B4, G4)$$

will be a stable matching whatever the unspecified preferences may be.

(b) Explain why the stable matching above is neither boy-optimal nor boy-pessimal and so will not be an outcome of the Mating Ritual.

(c) Describe how to define a set of marriage preferences among n boys and n girls which have at least $2^{n/2}$ stable assignments.

Hint: Arrange the boys into a list of $n/2$ pairs, and likewise arrange the girls into a list of $n/2$ pairs of girls. Choose preferences so that the kth pair of boys ranks the kth pair of girls just below the previous pairs of girls, and likewise for the kth pair of girls. Within the kth pairs, make sure each boy's first choice girl in the pair prefers the other boy in the pair.

Homework Problems

Problem 9.18.
The most famous application of stable matching was in assigning graduating medical students to hospital residencies. Each hospital has a preference ranking of students and each student has a preference order of hospitals, but unlike the setup in the notes where there are an equal number of boys and girls and monogamous marriages, hospitals generally have differing numbers of available residencies, and the total number of residencies may not equal the number of graduating students. Modify the definition of stable matching so it applies in this situation, and explain how to modify the Mating Ritual so it yields stable assignments of students to residencies. No proof is required.

Problem 9.19.
Give an example of a stable matching between 3 boys and 3 girls where no person gets their first choice. Briefly explain why your matching is stable.

Problem 9.20.
In a stable matching between n boys and girls produced by the Mating Ritual, call a person *lucky* if they are matched up with one of their $\lceil n/2 \rceil$ top choices. We will prove:

Theorem. *There must be at least one lucky person.*

To prove this, define the following derived variables for the Mating Ritual:

$q(B) = j$, where j is the rank of the girl that boy B is courting. That is to say, boy B is always courting the jth girl on his list.

$r(G)$ is the number of boys that girl G has rejected.

(a) Let

$$S ::= \sum_{B \in \text{the-Boys}} q(B) - \sum_{G \in \text{the-Girls}} r(G). \tag{9.7}$$

Show that S remains the same from one day to the next in the Mating Ritual.

(b) Prove the Theorem above. (You may assume for simplicity that n is even.)

Hint: A girl is sure to be lucky if she has rejected half the boys.