Chapter 8

Directed graphs

8.1 Digraphs

A *directed graph* (*digraph* for short) is formally the same as a binary relation, R, on a set, A —that is, a relation whose domain and codomain are the same set, A. But we describe digraphs as though they were diagrams, with elements of A pictured as points on the plane and arrows drawn between related points. The elements of A are referred to as the *vertices* of the digraph, and the pairs $(a, b) \in \text{graph}(R)$ are *directed edges*. Writing $a \to b$ is a more suggestive alternative for the pair (a, b). Directed edges are also called *arrows*.

For example, the divisibility relation on $\{1, 2, \ldots, 12\}$ is could be pictured by the digraph:

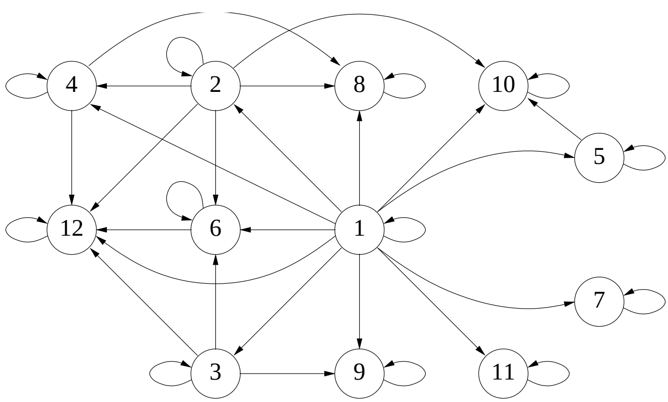


Figure 8.1: The Digraph for Divisibility on $\{1, 2, \ldots, 12\}$.

8.1.1 Paths in Digraphs

Picturing digraphs with points and arrows makes it natural to talk about following a *path* of successive edges through the graph. For example, in the digraph of Figure 8.1, a path might start at vertex 1, successively follow the edges from vertex 1 to vertex 2, from 2 to 4, from 4 to 12, and then from 12 to 12 twice (or as many times as you like). We can represent the path with the sequence of sucessive vertices it went through, in this case:

$$1, 2, 4, 12, 12, 12.$$

So a path is just a sequence of vertices, with consecutive vertices on the path connected by directed edges. Here is a formal definition:

Definition 8.1.1. A *path in a digraph* is a sequence of vertices $a_0, \ldots, a_k$ with $k \geq 0$ such that $a_i \to a_{i+1}$ is an edge of the digraph for $i = 0, 1, \ldots, k-1$. The path is said to *start* at a_0, to *end* at a_k, and the *length* of the path is defined to be k. The path is *simple* iff all the a_i's are different, that is, if $i \neq j$, then $a_i \neq a_j$.

Note that a single vertex counts as length zero path that begins and ends at itself.

It's pretty natural to talk about the edges in a path, but technically, paths only have points, not edges. So to instead, we'll say a path *traverses* an edge $a \to b$ when a and b are consecutive vertices in the path.

For any digraph, R, we can define some new relations on vertices based on paths, namely, the *path relation*, R^*, and the *positive-length path relation*, R^+:

$$a\ R^*\ b ::= \text{there is a path in } R \text{ from } a \text{ to } b,$$
$$a\ R^+\ b ::= \text{there is a positive length path in } R \text{ from } a \text{ to } b.$$

By the definition of path, both R^* and R^+ are transitive. Since edges count as length one paths, the edges of R^+ include all the edges of R. The edges of R^* in turn include all the edges of R^+ and, in addition include an edge (self-loop) from each vertex to itself. The self-loops get included in R^* because of the a length zero paths in R. So R^* is reflexive. [1]

8.2 Picturing Relational Properties

Many of the relational properties we've discussed have natural descriptions in terms of paths. For example:

Reflexivity: All vertices have self-loops (a *self-loop* at a vertex is an arrow going from the vertex back to itself).

Irreflexivity: No vertices have self-loops.

Antisymmetry: At most one (directed) edge between different vertices.

[1] In many texts, R^+ is called the *transitive closure* and R^* is called the *reflexive transitive closure* of R.

Asymmetry: No self-loops and at most one (directed) edge between different vertices.

Transitivity: Short-circuits—for any path through the graph, there is an arrow from the first vertex to the last vertex on the path.

Symmetry: A binary relation R is *symmetric* iff aRb implies bRa for all a, b in the domain of R. That is, if there is an edge from a to b, there is also one in the reverse direction.

8.3 Composition of Relations

There is a simple way to extend composition of functions to composition of relations, and this gives another way to talk about paths in digraphs.

Let $R : B \to C$ and $S : A \to B$ be relations. Then the composition of R with S is the binary relation $(R \circ S) : A \to C$ defined by the rule

$$a\ (R \circ S)\ c ::= \exists b \in B.\ (b\ R\ c) \text{ AND } (a\ S\ b).$$

This agrees with the Definition 4.3.1 of composition in the special case when R and S are functions.

Now when R is a digraph, it makes sense to compose R with itself. Then if we let R^n denote the composition of R with itself n times, it's easy to check that R^n is the length-n path relation:

$$a\ R^n\ b \quad \text{iff} \quad \text{there is a length } n \text{ path in } R \text{ from } a \text{ to } b.$$

This even works for $n = 0$, if we adopt the convention that R^0 is the identity relation Id_A on the set, A, of vertices. That is, $(a\ \text{Id}_A\ b)$ iff $a = b$.

8.4 Directed Acyclic Graphs

Definition 8.4.1. A *cycle* in a digraph is defined by a path that begins and ends at the same vertex. This includes the cycle of length zero that begins and ends at the vertex. A *directed acyclic graph (DAG)* is a directed graph with no *positive* length cycles.

A *simple cycle* in a digraph is a cycle whose vertices are distinct except for the beginning and end vertices.

DAG's can be an economical way to represent partial orders. For example, the *direct prerequisite* relation between MIT subjects in Chapter 7 was used to determine the partial order of indirect prerequisites on subjects. This indirect prerequisite partial order is precisely the positive length path relation of the direct prerequisites.

Lemma 8.4.2. *If D is a DAG, then D^+ is a strict partial order.*

Proof. We know that D^+ is transitive. Also, a positive length path from a vertex to itself would be a cycle, so there are no such paths. This means D^+ is irreflexive, which implies it is a strict partial order (see problem 7.8). ■

It's easy to check that conversely, the graph of any strict partial order is a DAG.

The divisibility partial order can also be more economically represented by the path relation in a DAG. A DAG whose *path* relation is divisibility on $\{1, 2, \ldots, 12\}$ is shown in Figure 8.2; the arrowheads are omitted in the Figure, and edges are understood to point upwards.

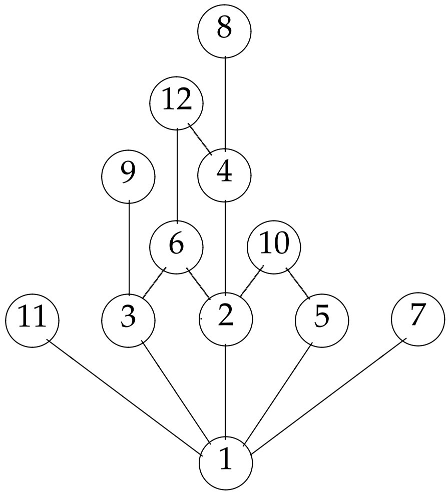


Figure 8.2: A DAG whose Path Relation is Divisibility on $\{1, 2, \ldots, 12\}$.

If we're using a DAG to represent a partial order —so all we care about is the the path relation of the DAG —we could replace the DAG with any other DAG with the same path relation. This raises the question of finding a DAG with the same path relation but the *smallest* number of edges. This DAG turns out to be unique and easy to find (see problem 8.2).

8.4.1 Problems

Practice Problems

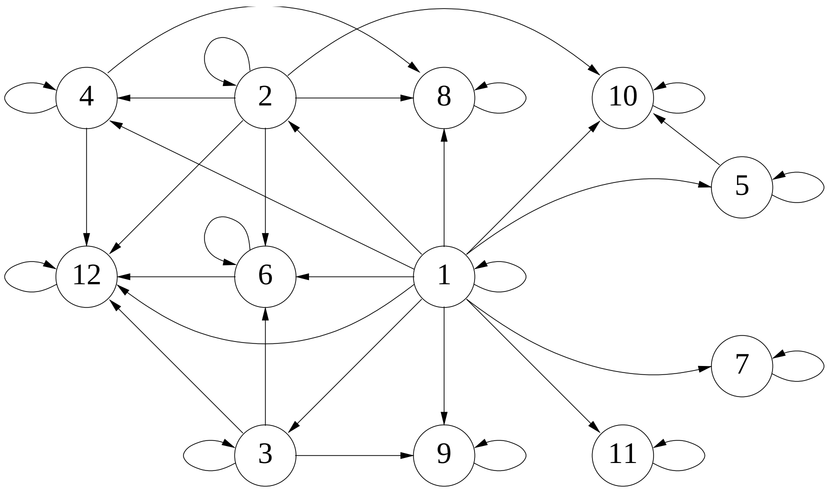
Problem 8.1.
Why is every strict partial order a DAG?

Class Problems

Problem 8.2.
If a and b are distinct nodes of a digraph, then a is said to *cover* b if there is an edge

from a to b and every path from a to b traverses this edge. If a covers b, the edge from a to b is called a *covering edge*.

(a) What are the covering edges in the following DAG?



(b) Let covering (D) be the subgraph of D consisting of only the covering edges. Suppose D is a finite DAG. Explain why covering (D) has the same positive path relation as D.

Hint: Consider *longest* paths between a pair of vertices.

(c) Show that if two *DAG*'s have the same positive path relation, then they have the same set of covering edges.

(d) Conclude that covering (D) is the *unique* DAG with the smallest number of edges among all digraphs with the same positive path relation as D.

The following examples show that the above results don't work in general for digraphs with cycles.

(e) Describe two graphs with vertices $\{1, 2\}$ which have the same set of covering edges, but not the same positive path relation (*Hint:* Self-loops.)

(f) (i) The *complete digraph* without self-loops on vertices $1, 2, 3$ has edges between every two distinct vertices. What are its covering edges?

(ii) What are the covering edges of the graph with vertices $1, 2, 3$ and edges $1 \to 2, 2 \to 3, 3 \to 1$?

(iii) What about their positive path relations?

Homework Problems

Problem 8.3.

Let R be a binary relation on a set A. Then R^n denotes the composition of R with itself n times. Let G_R be the digraph associated with R. That is, A is the set of vertices ofG_R and R is the set of directed edges. Let $R^{(n)}$ denote the length n path relation G_R, that is,

$$a\ R^{(n)}\ b ::= \text{there is a length } n \text{ path from } a \text{ to } b \text{ in } G_R.$$

Prove that

$$R^n = R^{(n)} \tag{8.1}$$

for all $n \in \mathbb{N}$.

Problem 8.4. **(a)** Prove that if R is a relation on a finite set, A, then

$$a\ (R \cup I_A)^n\ b \quad \text{iff} \quad \text{there is a path in } R \text{ of length length} \leq n \text{ from } a \text{ to } b.$$

(b) Conclude that if A is a finite set, then

$$R^* = (R \cup I_A)^{|A|-1}. \tag{8.2}$$

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