Chapter 1

What is a Proof?

1.1 Mathematical Proofs

A proof is a method of establishing truth. What constitutes a proof differs among fields.

- *Legal* truth is decided by a jury based on allowable evidence presented at trial.
- *Authoritative* truth is specified by a trusted person or organization.
- *Scientific* truth[1] is confirmed by experiment.
- *Probable* truth is established by statistical analysis of sample data.
- *Philosophical* proof involves careful exposition and persuasion typically based on a series of small, plausible arguments. The best example begins with "Cogito ergo sum," a Latin sentence that translates as "I think, therefore I am." It comes from the beginning of a 17th century essay by the mathematician/philospher, René Descartes, and it is one of the most famous quotes in the world: do a web search on the phrase and you will be flooded with hits.

  Deducing your existence from the fact that you're thinking about your existence is a pretty cool and persuasive-sounding first axiom. However, with just a few more lines of argument in this vein, Descartes goes on to conclude that there is an infinitely beneficent God. Whether or not you believe in a beneficent God, you'll probably agree that any very short proof of God's existence is bound to be far-fetched. So even in masterful hands, this approach is not reliable.

[1] Actually, only scientific *falsehood* can be demonstrated by an experiment —when the experiment fails to behave as predicted. But no amount of experiment can confirm that the *next* experiment won't fail. For this reason, scientists rarely speak of truth, but rather of *theories* that accurately predict past, and anticipated future, experiments.

Mathematics also has a specific notion of "proof."

Definition. A *formal proof* of a *proposition* is a chain of *logical deductions* leading to the proposition from a base set of *axioms*.

The three key ideas in this definition are highlighted: proposition, logical deduction, and axiom. In the next sections, we'll discuss these three ideas along with some basic ways of organizing proofs.

1.1.1 Problems

Class Problems

Problem 1.1.
Identify exactly where the bugs are in each of the following bogus proofs.[2]

(a) **Bogus Claim**: $1/8 > 1/4$.

Bogus proof.

$$\begin{gathered}3 > 2\\ 3\log_{10}(1/2) > 2\log_{10}(1/2)\\ \log_{10}(1/2)^3 > \log_{10}(1/2)^2\\ (1/2)^3 > (1/2)^2,\end{gathered}$$

and the claim now follows by the rules for multiplying fractions. ■

(b) *Bogus proof*: $1¢ = \$0.01 = (\$0.1)^2 = (10¢)^2 = 100¢ = \$1.$ ■

(c) **Bogus Claim**: If a and b are two equal real numbers, then $a = 0$.

Bogus proof.

$$\begin{aligned}a &= b\\ a^2 &= ab\\ a^2 - b^2 &= ab - b^2\\ (a-b)(a+b) &= (a-b)b\\ a+b &= b\\ a &= 0.\end{aligned}$$

■

[2] From Stueben, Michael and Diane Sandford. *Twenty Years Before the Blackboard*, Mathematical Association of America, ©1998.

Problem 1.2.
It's a fact that the Arithmetic Mean is at least as large the Geometric Mean, namely,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

for all nonnegative real numbers a and b. But there's something objectionable about the following proof of this fact. What's the objection, and how would you fix it?

Bogus proof.

$$\begin{aligned} \frac{a+b}{2} &\overset{?}{\geq} \sqrt{ab}, && \text{so} \\ a+b &\overset{?}{\geq} 2\sqrt{ab}, && \text{so} \\ a^2+2ab+b^2 &\overset{?}{\geq} 4ab, && \text{so} \\ a^2-2ab+b^2 &\overset{?}{\geq} 0, && \text{so} \\ (a-b)^2 &\geq 0 && \text{which we know is true.} \end{aligned}$$

The last statement is true because $a-b$ is a real number, and the square of a real number is never negative. This proves the claim. ■

Problem 1.3.
Albert announces that he plans a surprise 6.042 quiz next week. His students wonder if the quiz could be next Friday. The students realize that it obviously cannot, because if it hadn't been given before Friday, everyone would know that there was only Friday left on which to give it, so it wouldn't be a surprise any more.

So the students ask whether Albert could give the surprise quiz Thursday? They observe that if the quiz wasn't given *before* Thursday, it would have to be given *on* the Thursday, since they already know it can't be given on Friday. But having figured that out, it wouldn't be a surprise if the quiz was on Thursday either. Similarly, the students reason that the quiz can't be on Wednesday, Tuesday, or Monday. Namely, it's impossible for Albert to give a surprise quiz next week. All the students now relax, having concluded that Albert must have been bluffing.

And since no one expects the quiz, that's why, when Albert gives it on Tuesday next week, it really is a surprise!

What do you think is wrong with the students' reasoning?

1.2 Propositions

Definition. A *proposition* is a statement that is either true or false.

This definition sounds very general, but it does exclude sentences such as, "Wherefore art thou Romeo?" and "Give me an A!". But not all propositions are mathematical. For example, "Albert's wife's name is 'Irene' " happens to be true, and could be proved with legal documents and testimony of their children, but it's not a mathematical statement.

Mathematically meaningful propositions must be about well-defined mathematical objects like numbers, sets, functions, relations, etc., and they must be stated using mathematically precise language. We can illustrate this with a few examples.

Proposition 1.2.1. *2 + 3 = 5.*

This proposition is true.

A *prime* is an integer greater than one that is not divisible by any integer greater than 1 besides itself, for example, 2, 3, 5, 7, 11, ...

Proposition 1.2.2. *For every nonnegative integer, n, the value of $n^2 + n + 41$ is prime.*

Let's try some numerical experimentation to check this proposition. Let [3]

$$p(n) ::= n^2 + n + 41. \tag{1.1}$$

We begin with $p(0) = 41$ which is prime. $p(1) = 43$ which is prime. $p(2) = 47$ which is prime. $p(3) = 53$ which is prime. . . . $p(20) = 461$ which is prime. Hmmm, starts to look like a plausible claim. In fact we can keep checking through $n = 39$ and confirm that $p(39) = 1601$ is prime.

But $p(40) = 40^2 + 40 + 41 = 41 \cdot 41$, which is not prime. So it's not true that the expression is prime *for all* nonnegative integers. In fact, it's not hard to show that *no* nonconstant polynomial with integer coefficients can map all natural numbers into prime numbers. The point is that in general you can't check a claim about an infinite set by checking a finite set of its elements, no matter how large the finite set.

By the way, propositions like this about *all* numbers or other things are so common that there is a special notation for it. With this notation, Proposition 1.2.2 would be

$$\forall n \in \mathbb{N}.\ p(n) \text{ is prime.} \tag{1.2}$$

Here the symbol $\forall$ is read "for all". The symbol $\mathbb{N}$ stands for the set of *nonnegative integers*, namely, 0, 1, 2, 3, . . . (ask your TA for the complete list). The symbol "$\in$" is read as "is a member of" or simply as "is in". The period after the $\mathbb{N}$ is just a separator between phrases.

Here are two even more extreme examples:

Proposition 1.2.3. *$a^4 + b^4 + c^4 = d^4$ has no solution when a, b, c, d are positive integers.*

Euler (pronounced "oiler") conjectured this in 1769. But the proposition was proven false 218 years later by Noam Elkies at a liberal arts school up Mass Ave. The solution he found was $a = 95800, b = 217519, c = 414560, d = 422481$.

[3] The symbol ::= means "equal by definition." It's always ok to simply write "=" instead of ::=, but reminding the reader that an equality holds by definition can be helpful.

In logical notation, Proposition 1.2.3 could be written,

$$\forall a \in \mathbb{Z}^+ \, \forall b \in \mathbb{Z}^+ \, \forall c \in \mathbb{Z}^+ \, \forall d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4.$$

Here, $\mathbb{Z}^+$ is a symbol for the positive integers. Strings of $\forall$'s like this are usually abbreviated for easier reading:

$$\forall a, b, c, d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4.$$

Proposition 1.2.4. $313(x^3 + y^3) = z^3$ *has no solution when* $x, y, z \in \mathbb{Z}^+$.

This proposition is also false, but the smallest counterexample has more than 1000 digits!

Proposition 1.2.5. *Every map can be colored with 4 colors so that adjacent*[4] *regions have different colors.*

This proposition is true and is known as the "*Four-Color Theorem*". However, there have been many incorrect proofs, including one that stood for 10 years in the late 19th century before the mistake was found. An extremely laborious proof was finally found in 1976 by mathematicians Appel and Haken, who used a complex computer program to categorize the four-colorable maps; the program left a couple of thousand maps uncategorized, and these were checked by hand by Haken and his assistants—including his 15-year-old daughter. There was a lot of debate about whether this was a legitimate proof: the proof was too big to be checked without a computer, and no one could guarantee that the computer calculated correctly, nor did anyone have the energy to recheck the four-colorings of thousands of maps that were done by hand. Finally, about five years ago, a mostly intelligible proof of the Four-Color Theorem was found, though a computer is still needed to check colorability of several hundred special maps (see
http://www.math.gatech.edu/~thomas/FC/fourcolor.html). [5]

Proposition 1.2.6 (Goldbach)**.** *Every even integer greater than 2 is the sum of two primes.*

No one knows whether this proposition is true or false. It is known as *Goldbach's Conjecture*, and dates back to 1742.

For a computer scientist, some of the most important things to prove are the "correctness" programs and systems —whether a program or system does what it's supposed to. Programs are notoriously buggy, and there's a growing community of researchers and practitioners trying to find ways to prove program correctness. These efforts have been successful enough in the case of CPU chips that they are now routinely used by leading chip manufacturers to prove chip correctness and avoid mistakes like the notorious Intel division bug in the 1990's.

Developing mathematical methods to verify programs and systems remains an active research area. We'll consider some of these methods later in the course.

[4] Two regions are adjacent only when they share a boundary segment of positive length. They are not considered to be adjacent if their boundaries meet only at a few points.

[5] The story of the Four-Color Proof is told in a well-reviewed popular (non-technical) book: "Four Colors Suffice. How the Map Problem was Solved." *Robin Wilson*. Princeton Univ. Press, 2003, 276pp. ISBN 0-691-11533-8.

1.3 Predicates

A *predicate* is a proposition whose truth depends on the value of one or more variables. Most of the propostions above were defined in terms of predicates. For example,

$$\text{“}n \text{ is a perfect square”}$$

is a predicate whose truth depends on the value of n. The predicate is true for $n = 4$ since four is a perfect square, but false for $n = 5$ since five is not a perfect square.

Like other propositions, predicates are often named with a letter. Furthermore, a function-like notation is used to denote a predicate supplied with specific variable values. For example, we might name our earlier predicate P:

$$P(n) ::= \text{“}n \text{ is a perfect square”}$$

Now $P(4)$ is true, and $P(5)$ is false.

This notation for predicates is confusingly similar to ordinary function notation. If P is a predicate, then $P(n)$ is either *true* or *false*, depending on the value of n. On the other hand, if p is an ordinary function, like $n^2 + 1$, then $p(n)$ is a *numerical quantity*. **Don't confuse these two!**

1.4 The Axiomatic Method

The standard procedure for establishing truth in mathematics was invented by Euclid, a mathematician working in Alexandria, Egypt around 300 BC. His idea was to begin with five *assumptions* about geometry, which seemed undeniable based on direct experience. (For example, “There is a straight line segment between every pair of points.) Propositions like these that are simply accepted as true are called *axioms*.

Starting from these axioms, Euclid established the truth of many additional propositions by providing “proofs”. A *proof* is a sequence of logical deductions from axioms and previously-proved statements that concludes with the proposition in question. You probably wrote many proofs in high school geometry class, and you'll see a lot more in this course.

There are several common terms for a proposition that has been proved. The different terms hint at the role of the proposition within a larger body of work.

- Important propositions are called *theorems*.
- A *lemma* is a preliminary proposition useful for proving later propositions.
- A *corollary* is a proposition that follows in just a few logical steps from a theorem.

The definitions are not precise. In fact, sometimes a good lemma turns out to be far more important than the theorem it was originally used to prove.

Euclid's axiom-and-proof approach, now called the *axiomatic method*, is the foundation for mathematics today. In fact, just a handful of axioms, called the axioms Zermelo-Frankel with Choice (*ZFC*), together with a few logical deduction rules, appear to be sufficient to derive essentially all of mathematics. We'll examine these in Chapter 4.

1.5 Our Axioms

The ZFC axioms are important in studying and justifying the foundations of mathematics, but for practical purposes, they are much too primitive. Proving theorems in ZFC is a little like writing programs in byte code instead of a full-fledged programming language —by one reckoning, a formal proof in ZFC that $2 + 2 = 4$ requires more than 20,000 steps! So instead of starting with ZFC, we're going to take a *huge* set of axioms as our foundation: we'll accept all familiar facts from high school math!

This will give us a quick launch, but you may find this imprecise specification of the axioms troubling at times. For example, in the midst of a proof, you may find yourself wondering, "Must I prove this little fact or can I take it as an axiom?" Feel free to ask for guidance, but really there is no absolute answer. Just be up front about what you're assuming, and don't try to evade homework and exam problems by declaring everything an axiom!

1.5.1 Logical Deductions

Logical deductions or *inference rules* are used to prove new propositions using previously proved ones.

A fundamental inference rule is *modus ponens*. This rule says that a proof of P together with a proof that P IMPLIES Q is a proof of Q.

Inference rules are sometimes written in a funny notation. For example, *modus ponens* is written:

Rule.

$$\frac{P, \quad P \text{ IMPLIES } Q}{Q}$$

When the statements above the line, called the *antecedents*, are proved, then we can consider the statement below the line, called the *conclusion* or *consequent*, to also be proved.

A key requirement of an inference rule is that it must be *sound*: any assignment of truth values that makes all the antecedents true must also make the consequent true. So if we start off with true axioms and apply sound inference rules, everything we prove will also be true.

There are many other natural, sound inference rules, for example:

Rule.

$$\frac{P \text{ IMPLIES } Q, \quad Q \text{ IMPLIES } R}{P \text{ IMPLIES } R}$$

Rule.

$$\frac{\text{NOT}(P) \text{ IMPLIES } \text{NOT}(Q)}{Q \text{ IMPLIES } P}$$

On the other hand,

Rule.

$$\frac{\text{NOT}(P) \text{ IMPLIES } \text{NOT}(Q)}{P \text{ IMPLIES } Q}$$

is not sound: if P is assigned **T** and Q is assigned **F**, then the antecedent is true and the consequent is not.

Note that a propositional inference rule is sound precisely when the conjunction (AND) of all its antecedents implies its consequent.

As with axioms, we will not be too formal about the set of legal inference rules. Each step in a proof should be clear and "logical"; in particular, you should state what previously proved facts are used to derive each new conclusion.

1.5.2 Patterns of Proof

In principle, a proof can be *any* sequence of logical deductions from axioms and previously proved statements that concludes with the proposition in question. This freedom in constructing a proof can seem overwhelming at first. How do you even *start* a proof?

Here's the good news: many proofs follow one of a handful of standard templates. Each proof has it own details, of course, but these templates at least provide you with an outline to fill in. We'll go through several of these standard patterns, pointing out the basic idea and common pitfalls and giving some examples. Many of these templates fit together; one may give you a top-level outline while others help you at the next level of detail. And we'll show you other, more sophisticated proof techniques later on.

The recipes below are very specific at times, telling you exactly which words to write down on your piece of paper. You're certainly free to say things your own way instead; we're just giving you something you *could* say so that you're never at a complete loss.

1.6 Proving an Implication

Propositions of the form "If P, then Q" are called *implications*. This implication is often rephrased as "P IMPLIES Q."

Here are some examples:

- (Quadratic Formula) If $ax^2 + bx + c = 0$ and $a \neq 0$, then
$$x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a.$$
- (Goldbach's Conjecture) If n is an even integer greater than 2, then n is a sum of two primes.
- If $0 \leq x \leq 2$, then $-x^3 + 4x + 1 > 0$.

There are a couple of standard methods for proving an implication.

1.6.1 Method #1

In order to prove that P IMPLIES Q:

1. Write, "Assume P."
2. Show that Q logically follows.

Example

Theorem 1.6.1. *If $0 \leq x \leq 2$, then $-x^3 + 4x + 1 > 0$.*

Before we write a proof of this theorem, we have to do some scratchwork to figure out why it is true.

The inequality certainly holds for $x = 0$; then the left side is equal to 1 and $1 > 0$. As x grows, the $4x$ term (which is positive) initially seems to have greater magnitude than $-x^3$ (which is negative). For example, when $x = 1$, we have $4x = 4$, but $-x^3 = -1$ only. In fact, it looks like $-x^3$ doesn't begin to dominate until $x > 2$. So it seems the $-x^3 + 4x$ part should be nonnegative for all x between 0 and 2, which would imply that $-x^3 + 4x + 1$ is positive.

So far, so good. But we still have to replace all those "seems like" phrases with solid, logical arguments. We can get a better handle on the critical $-x^3 + 4x$ part by factoring it, which is not too hard:

$$-x^3 + 4x = x(2 - x)(2 + x)$$

Aha! For x between 0 and 2, all of the terms on the right side are nonnegative. And a product of nonnegative terms is also nonnegative. Let's organize this blizzard of observations into a clean proof.

Proof. Assume $0 \leq x \leq 2$. Then x, $2 - x$, and $2 + x$ are all nonnegative. Therefore, the product of these terms is also nonnegative. Adding 1 to this product gives a positive number, so:

$$x(2 - x)(2 + x) + 1 > 0$$

Multiplying out on the left side proves that

$$-x^3 + 4x + 1 > 0$$

as claimed. ■

There are a couple points here that apply to all proofs:

- You'll often need to do some scratchwork while you're trying to figure out the logical steps of a proof. Your scratchwork can be as disorganized as you like— full of dead-ends, strange diagrams, obscene words, whatever. But keep your scratchwork separate from your final proof, which should be clear and concise.

- Proofs typically begin with the word "Proof" and end with some sort of doohickey like □ or "q.e.d". The only purpose for these conventions is to clarify where proofs begin and end.

1.6.2 Method #2 - Prove the Contrapositive

An implication ("P IMPLIES Q") is logically equivalent to its *contrapositive*

$$\text{NOT}(Q) \text{ IMPLIES NOT}(P)$$

Proving one is as good as proving the other, and proving the contrapositive is sometimes easier than proving the original statement. If so, then you can proceed as follows:

1. Write, "We prove the contrapositive:" and then state the contrapositive.

2. Proceed as in Method #1.

Example

Theorem 1.6.2. *If r is irrational, then $\sqrt{r}$ is also irrational.*

Recall that rational numbers are equal to a ratio of integers and irrational numbers are not. So we must show that if r is *not* a ratio of integers, then $\sqrt{r}$ is also *not* a ratio of integers. That's pretty convoluted! We can eliminate both *not*'s and make the proof straightforward by considering the contrapositive instead.

Proof. We prove the contrapositive: if $\sqrt{r}$ is rational, then r is rational.

Assume that $\sqrt{r}$ is rational. Then there exist integers a and b such that:

$$\sqrt{r} = \frac{a}{b}$$

Squaring both sides gives:

$$r = \frac{a^2}{b^2}$$

Since a^2 and b^2 are integers, r is also rational. ■

1.6.3 Problems

Homework Problems

Problem 1.4.
Show that $\log_7 n$ is either an integer or irrational, where n is a positive integer. Use whatever familiar facts about integers and primes you need, but explicitly state such facts. (This problem will be graded on the clarity and simplicity of your proof. If you can't figure out how to prove it, ask the staff for help and they'll tell you how.)

1.7 Proving an "If and Only If"

Many mathematical theorems assert that two statements are logically equivalent; that is, one holds if and only if the other does. Here is an example that has been known for several thousand years:

> Two triangles have the same side lengths if and only if two side lengths and the angle between those sides are the same.

The phrase "if and only if" comes up so often that it is often abbreviated "iff".

1.7.1 Method #1: Prove Each Statement Implies the Other

The statement "P IFF Q" is equivalent to the two statements "P IMPLIES Q" and "Q IMPLIES P". So you can prove an "iff" by proving *two* implications:

1. Write, "We prove P implies Q and vice-versa."

2. Write, "First, we show P implies Q." Do this by one of the methods in Section 1.6.

3. Write, "Now, we show Q implies P." Again, do this by one of the methods in Section 1.6.

1.7.2 Method #2: Construct a Chain of Iffs

In order to prove that P is true iff Q is true:

1. Write, "We construct a chain of if-and-only-if implications."

2. Prove P is equivalent to a second statement which is equivalent to a third statement and so forth until you reach Q.

This method sometimes requires more ingenuity than the first, but the result can be a short, elegant proof.

Example

The *standard deviation* of a sequence of values $x_1, x_2, \ldots, x_n$ is defined to be:

$$\sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_n-\mu)^2}{n}} \tag{1.3}$$

where μ is the *mean* of the values:

$$\mu ::= \frac{x_1+x_2+\cdots+x_n}{n}$$

Theorem 1.7.1. *The standard deviation of a sequence of values $x_1, \ldots, x_n$ is zero iff all the values are equal to the mean.*

For example, the standard deviation of test scores is zero if and only if everyone scored exactly the class average.

Proof. We construct a chain of "iff" implications, starting with the statement that the standard deviation (1.3) is zero:

$$\sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_n-\mu)^2}{n}} = 0. \tag{1.4}$$

Now since zero is the only number whose square root is zero, equation (1.4) holds iff

$$(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_n-\mu)^2 = 0. \tag{1.5}$$

Now squares of real numbers are always nonnegative, so every term on the left hand side of equation (1.5) is nonnegative. This means that (1.5) holds iff

$$\text{Every term on the left hand side of (1.5) is zero.} \tag{1.6}$$

But a term $(x_i-\mu)^2$ is zero iff $x_i = \mu$, so (1.6) is true iff

$$\text{Every } x_i \text{ equals the mean.}$$

■

1.8 Proof by Cases

Breaking a complicated proof into cases and proving each case separately is a useful, common proof strategy. Here's an amusing example.

Let's agree that given any two people, either they have met or not. If every pair of people in a group has met, we'll call the group a *club*. If every pair of people in a group has not met, we'll call it a group of *strangers*.

Theorem. *Every collection of 6 people includes a club of 3 people or a group of 3 strangers.*

Proof. The proof is by case analysis[6]. Let x denote one of the six people. There are two cases:

1. Among 5 other people besides x, at least 3 have met x.
2. Among the 5 other people, at least 3 have not met x.

Now we have to be sure that at least one of these two cases must hold,[7] but that's easy: we've split the 5 people into two groups, those who have shaken hands with x and those who have not, so one the groups must have at least half the people.

Case 1: Suppose that at least 3 people did meet x.

This case splits into two subcases:

> **Case 1.1:** No pair among those people met each other. Then these people are a group of at least 3 strangers. So the Theorem holds in this subcase.
>
> **Case 1.2:** Some pair among those people have met each other. Then that pair, together with x, form a club of 3 people. So the Theorem holds in this subcase.

This implies that the Theorem holds in Case 1.

Case 2: Suppose that at least 3 people did not meet x.

This case also splits into two subcases:

> **Case 2.1**: Every pair among those people met each other. Then these people are a club of at least 3 people. So the Theorem holds in this subcase.
>
> **Case 2.2:** Some pair among those people have not met each other. Then that pair, together with x, form a group of at least 3 strangers. So the Theorem holds in this subcase.

This implies that the Theorem also holds in Case 2, and therefore holds in all cases. ■

1.8.1 Problems

Class Problems

Problem 1.5.

If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

[6] Describing your approach at the outset helps orient the reader.

[7] Part of a case analysis argument is showing that you've covered all the cases. Often this is obvious, because the two cases are of the form "P" and "not P". However, the situation above is not stated quite so simply.

Homework Problems

Problem 1.6.
For $n = 40$, the value of polynomial $p(n) ::= n^2 + n + 41$ is not prime, as noted in Chapter 1 of the Course Text. But we could have predicted based on general principles that no nonconstant polynomial, $q(n)$, with integer coefficients can map each nonnegative integer into a prime number. Prove it.

Hint: Let $c ::= q(0)$ be the constant term of q. Consider two cases: c is not prime, and c is prime. In the second case, note that $q(cn)$ is a multiple of c for all $n \in \mathbb{Z}$. You may assume the familiar fact that the magnitude (absolute value) of any nonconstant polynomial, $q(n)$, grows unboundedly as n grows.

1.9 Proof by Contradiction

In a *proof by contradiction* or *indirect proof*, you show that if a proposition were false, then some false fact would be true. Since a false fact can't be true, the proposition had better not be false. That is, the proposition really must be true.

Proof by contradiction is *always* a viable approach. However, as the name suggests, indirect proofs can be a little convoluted. So direct proofs are generally preferable as a matter of clarity.

Method: In order to prove a proposition P by contradiction:

1. Write, "We use proof by contradiction."

2. Write, "Suppose P is false."

3. Deduce something known to be false (a logical contradiction).

4. Write, "This is a contradiction. Therefore, P must be true."

Example

Remember that a number is *rational* if it is equal to a ratio of integers. For example, $3.5 = 7/2$ and $0.1111\cdots = 1/9$ are rational numbers. On the other hand, we'll prove by contradiction that $\sqrt{2}$ is irrational.

Theorem 1.9.1. *$\sqrt{2}$ is irrational.*

Proof. We use proof by contradiction. Suppose the claim is false; that is, $\sqrt{2}$ is rational. Then we can write $\sqrt{2}$ as a fraction n/d in *lowest terms*.

Squaring both sides gives $2 = n^2/d^2$ and so $2d^2 = n^2$. This implies that n is a multiple of 2. Therefore n^2 must be a multiple of 4. But since $2d^2 = n^2$, we know $2d^2$ is a multiple of 4 and so d^2 is a multiple of 2. This implies that d is a multiple of 2.

So the numerator and denominator have 2 as a common factor, which contradicts the fact that n/d is in lowest terms. So $\sqrt{2}$ must be irrational. ■

1.9.1 Problems

Class Problems

Problem 1.7.
Generalize the proof from lecture (reproduced below) that $\sqrt{2}$ is irrational, for example, how about $\sqrt[3]{2}$? Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

Theorem. *$\sqrt{2}$ is an irrational number.*

Proof. The proof is by contradiction: assume that $\sqrt{2}$ is rational, that is,

$$\sqrt{2} = \frac{n}{d}, \tag{1.7}$$

where n and d are integers. Now consider the smallest such positive integer denominator, d. We will prove in a moment that the numerator, n, and the denominator, d, are both even. This implies that

$$\frac{n/2}{d/2}$$

is a fraction equal to $\sqrt{2}$ with a smaller positive integer denominator, a contradiction.

> *Since the assumption that $\sqrt{2}$ is rational leads to this contradiction, the assumption must be false. That is, $\sqrt{2}$ is indeed irrational.* This italicized comment on the implication of the contradiction normally goes without saying, but since this is the first 6.042 exercise about proof by contradiction, we've said it.

To prove that n and d have 2 as a common factor, we start by squaring both sides of (1.7) and get $2 = n^2/d^2$, so

$$2d^2 = n^2. \tag{1.8}$$

So 2 is a factor of n^2, which is only possible if 2 is in fact a factor of n.

This means that $n = 2k$ for some integer, k, so

$$n^2 = (2k)^2 = 4k^2. \tag{1.9}$$

Combining (1.8) and (1.9) gives $2d^2 = 4k^2$, so

$$d^2 = 2k^2. \tag{1.10}$$

So 2 is a factor of d^2, which again is only possible if 2 is in fact also a factor of d, as claimed. ■

Problem 1.8.
Here is a different proof that $\sqrt{2}$ is irrational, taken from the American Mathematical Monthly, v.116, #1, Jan. 2009, p.69:

Proof. Suppose for the sake of contradiction that $\sqrt{2}$ is rational, and choose the least integer, $q > 0$, such that $\left(\sqrt{2}-1\right)q$ is a nonnegative integer. Let $q' ::= \left(\sqrt{2}-1\right)q$. Clearly $0 < q' < q$. But an easy computation shows that $\left(\sqrt{2}-1\right)q'$ is a nonnegative integer, contradicting the minimality of q. ■

(a) This proof was written for an audience of college teachers, and is a little more concise than desirable at this point in 6.042. Write out a more complete version which includes an explanation of each step.

(b) Now that you have justified the steps in this proof, do you have a preference for one of these proofs over the other? Why? Discuss these questions with your teammates for a few minutes and summarize your team's answers on your whiteboard.

Problem 1.9.
Here is a generalization of Problem 1.7 that you may not have thought of:

Lemma 1.9.2. *Let the coefficients of the polynomial $a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{m-1}+x^m$ be integers. Then any real root of the polynomial is either integral or irrational.*

(a) Explain why Lemma 1.9.2 immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an mth power of some integer.

(b) Collaborate with your tablemates to write a clear, textbook quality proof of Lemma 1.9.2 on your whiteboard. (Besides clarity and correctness, textbook quality requires good English with proper punctuation. When a real textbook writer does this, it usually takes multiple revisions; if you're satisfied with your first draft, you're probably misjudging.) You may find it helpful to appeal to the following:

Lemma 1.9.3. *If a prime, p, is a factor of some power of an integer, then it is a factor of that integer.*

You may assume Lemma 1.9.3 without writing down its proof, but see if you can explain why it is true.

Homework Problems

Problem 1.10.
The fact that that there are irrational numbers a, b such that a^b is rational was proved in Problem 1.5. Unfortunately, that proof was *nonconstructive*: it didn't reveal a specific pair, a, b, with this property. But in fact, it's easy to do this: let $a ::= \sqrt{2}$ and $b ::= 2\log_2 3$.

We know $\sqrt{2}$ is irrational, and obviously $a^b = 3$. Finish the proof that this a, b pair works, by showing that $2\log_2 3$ is irrational.

1.10 *Good* Proofs in Practice

One purpose of a proof is to establish the truth of an assertion with absolute certainty. Mechanically checkable proofs of enormous length or complexity can accomplish this. But humanly intelligible proofs are the only ones that help someone understand the subject. Mathematicians generally agree that important mathematical results can't be fully understood until their proofs are understood. That is why proofs are an important part of the curriculum.

To be understandable and helpful, more is required of a proof than just logical correctness: a good proof must also be clear. Correctness and clarity usually go together; a well-written proof is more likely to be a correct proof, since mistakes are harder to hide.

In practice, the notion of proof is a moving target. Proofs in a professional research journal are generally unintelligible to all but a few experts who know all the terminology and prior results used in the proof. Conversely, proofs in the first weeks of a beginning course like 6.042 would be regarded as tediously long-winded by a professional mathematician. In fact, what we accept as a good proof later in the term will be different from what we consider good proofs in the first couple of weeks of 6.042. But even so, we can offer some general tips on writing good proofs:

State your game plan. A good proof begins by explaining the general line of reasoning, for example, "We use case analysis" or "We argue by contradiction."

Keep a linear flow. Sometimes proofs are written like mathematical mosaics, with juicy tidbits of independent reasoning sprinkled throughout. This is not good. The steps of an argument should follow one another in an intelligble order.

A proof is an essay, not a calculation. Many students initially write proofs the way they compute integrals. The result is a long sequence of expressions without explanation, making it very hard to follow. This is bad. A good proof usually looks like an essay with some equations thrown in. Use complete sentences.

Avoid excessive symbolism. Your reader is probably good at understanding words, but much less skilled at reading arcane mathematical symbols. So use words where you reasonably can.

Revise and simplify. Your readers will be grateful.

Introduce notation thoughtfully. Sometimes an argument can be greatly simplified by introducing a variable, devising a special notation, or defining a new term. But do this sparingly since you're requiring the reader to remember all that new stuff. And remember to actually *define* the meanings of new variables, terms, or notations; don't just start using them!

Structure long proofs. Long programs are usually broken into a hierarchy of smaller procedures. Long proofs are much the same. Facts needed in your proof that are easily stated, but not readily proved are best pulled out and proved in preliminary lemmas. Also, if you are repeating essentially the same argument over and over, try to capture that argument in a general lemma, which you can cite repeatedly instead.

Be wary of the "obvious". When familiar or truly obvious facts are needed in a proof, it's OK to label them as such and to not prove them. But remember that what's obvious to you, may not be —and typically is not —obvious to your reader.

Most especially, don't use phrases like "clearly" or "obviously" in an attempt to bully the reader into accepting something you're having trouble proving. Also, go on the alert whenever you see one of these phrases in someone else's proof.

Finish. At some point in a proof, you'll have established all the essential facts you need. Resist the temptation to quit and leave the reader to draw the "obvious" conclusion. Instead, tie everything together yourself and explain why the original claim follows.

The analogy between good proofs and good programs extends beyond structure. The same rigorous thinking needed for proofs is essential in the design of critical computer systems. When algorithms and protocols only "mostly work" due to reliance on hand-waving arguments, the results can range from problematic to catastrophic. An early example was the Therac 25, a machine that provided radiation therapy to cancer victims, but occasionally killed them with massive overdoses due to a software race condition. A more recent (August 2004) example involved a single faulty command to a computer system used by United and American Airlines that grounded the entire fleet of both companies— and all their passengers!

It is a certainty that we'll all one day be at the mercy of critical computer systems designed by you and your classmates. So we really hope that you'll develop the ability to formulate rock-solid logical arguments that a system actually does what you think it does!

6.042J / 18.062J Mathematics for Computer Science
Spring 2010