Solutions to In-Class Problems Week 14, Wed.

Problem 1.
A gambler is placing $1 bets on the "1st dozen" in roulette. This bet wins when a number from one to twelve comes in, and then the gambler gets his $1 back plus $3 more. Recall that there are 38 numbers on the roulette wheel.

The gambler's initial stake in $\$n$ and his target is $\$T$. He will keep betting until he runs out of money ("goes broke") or reachs his target. Let w_n be the probability of the gambler winning, that is, reaching target $\$T$ before going broke.

(a) Write a linear recurrence for w_n; you need *not* solve the recurrence.

Solution. The probability of winning a bet is $12/38$. Thus, by the Law of Total Probability **??**,

$$\begin{aligned} w_n &= \Pr\{\text{win starting with } \$n \mid \text{won first bet}\} \cdot \Pr\{\text{won first bet}\} + \Pr\{\text{win starting with } \$n \mid \text{lost first bet}\} \cdot \Pr \\ &= \Pr\{\text{win starting with } \$n+3\} \cdot \Pr\{\text{won first bet}\} + \Pr\{\text{win starting with } \$n-1\} \cdot \Pr\{\text{lost first bet}\}, \end{aligned}$$

so

$$w_n = \frac{12}{38} w_{n+3} + \frac{26}{38} w_{n-1}.$$

Letting $m ::= n + 3$ we get

$$w_m = \frac{38}{12} w_{m-3} - \frac{26}{12} w_{m-4}.$$

■

(b) Let e_n be the expected number of bets until the game ends. Write a linear recurrence for e_n; you need *not* solve the recurrence.

Solution. By the Law of Total Expectation, Theorem **??**,

$$\begin{aligned} e_n &= (1 + \mathrm{E}[\text{number of bets starting with } \$n \mid \text{won first bet}]) \cdot \Pr\{\text{won first bet}\} + (1 + \mathrm{E}[\text{number of bets starting} \\ &= (1 + \mathrm{E}[\text{number of bets starting with } \$n+3]) \cdot \Pr\{\text{won first bet}\} + (1 + \mathrm{E}[\text{number of bets starting with } \$n - \end{aligned}$$

so

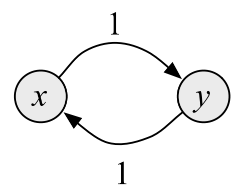
$$e_n = (e_{n+3} + 1)\frac{12}{38} + (1 + e_{n-1})\frac{26}{38}$$

Letting $m ::= n + 3$ we get

$$e_m = \frac{38}{12} e_{m-3} - \frac{1 - 26/12}{26/12} \cdot e_{m-4} - \frac{38}{12}$$

■

Problem 2.
Consider the following random-walk graph:



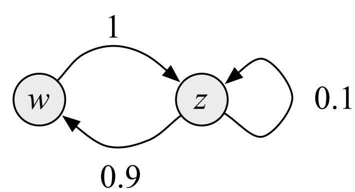
(a) Find a stationary distribution.

Solution. $d(x) = d(y) = 1/2$ ■

(b) If you start at node x and take a (long) random walk, does the distribution over nodes ever get close to the stationary distribution? Explain.

Solution. No! you just alternate between nodes x and y. ■

Consider the following random-walk graph:



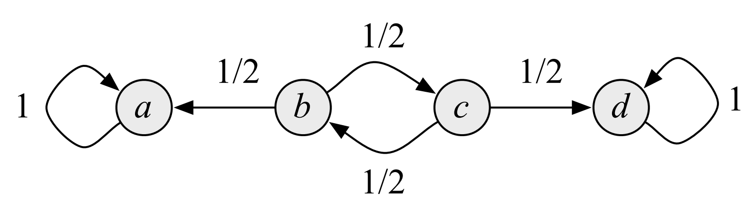
(c) Find a stationary distribution.

Solution. $d(w) = 9/19$, $d(z) = 10/19$. You can derive this by setting $d(w) = (9/10)d(z)$, $d(z) = d(w) + (1/10)d(z)$, and $d(w) + d(z) = 1$. There is a unique solution. ■

(d) If you start at node w and take a (long) random walk, does the distribution over nodes ever get close to the stationary distribution? We don't want you to prove anything here, just write out a few steps and see what's happening.

Solution. Yes, it does. ■

Consider the following random-walk graph:



(e) Describe the stationary distributions for this graph.

Solution. There are infinitely many, with $d(b) = d(c) = 0$, and $d(a) = p$ and $d(d) = 1 - p$ for any p. ■

(f) If you start at node b and take a long random walk, the probability you are at node d will be close to what fraction? Explain.

Solution. $1/3$. ■

Appendix

A *random-walk graph* is a digraph such that each edge, $x \to y$, is labelled with a number, $p(x, y) > 0$, which will indicate the probability of following that edge starting at vertex x. Formally, we simply require that the sum of labels leaving each vertex is 1. That is, if we define for each vertex, x,

$$\text{out}(x) ::= \{y \mid x \to y \text{ is an edge of the graph}\},$$

then

$$\sum_{y \in \text{out}(x)} p(x, y) = 1.$$

A *distribution*, d, is a labelling of each vertex, x, with a number, $d(x) \geq 0$, which will indicate the probability of being at x. Formally, we simply require that the sum of all the vertex labels is 1, that is,

$$\sum_{x \in V} d(x) = 1,$$

where V is the set of vertices.

The distribution, $\widehat{d}$, *after a single step* of a random walk from distribution, d, is given by

$$\widehat{d}(x) ::= \sum_{y \in \text{in}(x)} d(y) \cdot p(y, x),$$

where

$$\text{in}(x) ::= \{y \mid y \to x \text{ is an edge of the graph}\}.$$

A distribution d is *stationary* if $\widehat{d} = d$, where $\widehat{d}$ is the distribution after a single step of a random walk starting from d. In other words, d stationary implies

$$d(x) ::= \sum_{y \in \text{in}(x)} d(y) \cdot p(y, x).$$