*Mathematics for Computer Science*

MIT 6.042J/18.062J

# Random Walks

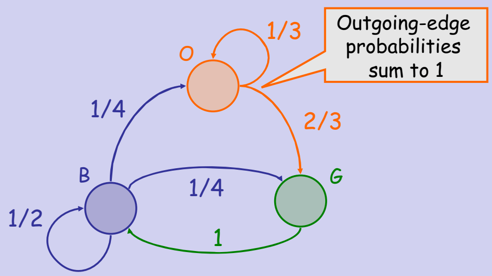


## Applications of Random Walk

- Physics — Brownian motion
- Finance — stocks, options
- Algorithms — web search, clustering

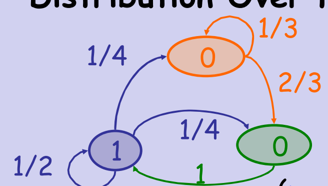


## Graph With Probable Transitions






## Distribution Over Nodes

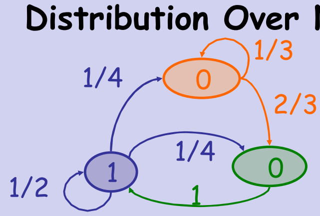



Suppose you start at B: $(p_B, p_O, p_G)$ = (1, 0, 0)

What are $p'_B$, $p'_O$, $p'_G$ after 1 step?



## Distribution Over Nodes

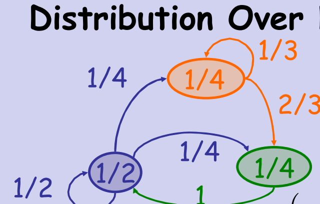



Dist after 1 step: $(p'_B, p'_G, p'_O)$

only get places from B, so $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$



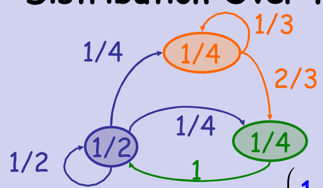
## Distribution Over Nodes




Dist after 1 step: $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

Dist after 2 steps: $(p''_B, p''_O, p''_G)$

## Distribution Over Nodes

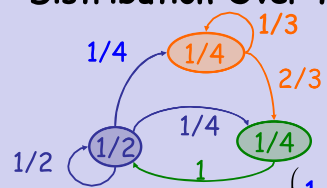



Dist after **1** step: $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

$p''_O$ = Pr{B to O | at B}·$p'_B$
+ Pr{O to O | at O}·$p'_O$
+ Pr{G to O | at G}·$p'_G$



## Distribution Over Nodes

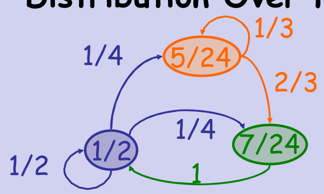



Dist after **1** step: $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

$p''_O$ = 1/4 · 1/2
+ 1/3 · 1/4
+ 0 · 1/4 = 5/24



## Distribution Over Nodes



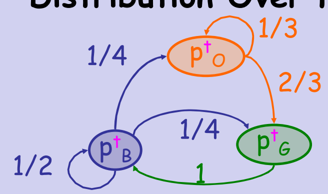

distribution after 2 steps:

$(p''_B, p''_O, p''_G)$

$\left(\frac{1}{2}, \frac{5}{24}, \frac{7}{24}\right)$



## Distribution Over Nodes

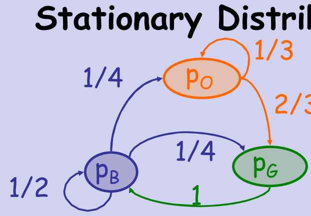



distribution after $t$ steps?

...and as $t \to \infty$?



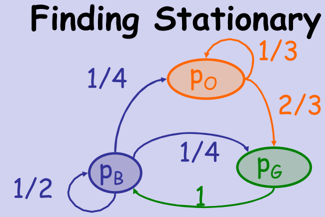
## Stationary Distribution




distribution $(p_B, p_O, p_G)$ is **stationary** if next-step distribution is the same.
What is a stationary dist. here?



## Finding Stationary Dist.



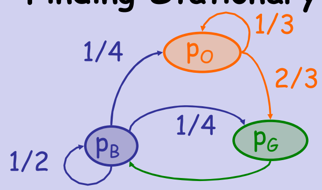

$p_B = p_B' = (1/2)p_B + 1p_G$

$p_O = p_O' = (1/4)p_B + (1/3)p_O$

$p_G = p_G' = (1/4)p_B + (2/3)p_O$

$p_B + p_O + p_G = 1$

## Finding Stationary Dist.




solving for $(p_B, p_O, p_G)$: $\left(\frac{8}{15}, \frac{3}{15}, \frac{4}{15}\right)$



## Google Page Rank

View the entire web as a graph

- vertices are webpages
- edge (u,v) exists if link from page u to page v
- Pr{go to v from u} = 1/outdeg(u)

Find stationary distribution $\{p_u\}$

Rank u above v if $p_u > p_v$.

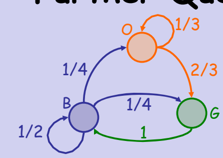


## Questions on Stationary Dist

- Does a stationary dist exist? Yes (if graph finite)
- Is it unique? Sometimes
- Does a random walk approach it from any starting distribution? Sometimes
  - How quickly? Varies



## Further Questions




- Pr{ever reach O | start at B}
- Pr{reach G before O | start at B}
- Average # steps for B to O



# Gambler's Ruin

(let's go to Vegas)



## Gambler's Ruin

- Decide to place \$1 bets until either going broke or reaching some target amount of money.
- What is Pr{reach target}?

## Gambling: Fair Case

Suppose we're playing a fair game:

- Pr{win bet} = 1/2.

What is Pr{reach $200} if we start with $100?

1/2

What about Pr{reach $600} if we start with $500?

5/6



## Gambling: Fair Case

In general, if we start with $n

Pr{reach $T} = n/T

What about an unfair game?



## Gambling: Slightly Unfair

Betting **black** in US roulette

Image by MIT OpenCourseWare.

Pr{win bet} = 18/38 = 9/19 < 1/2



## US Roulette

What is Pr{reach $500+100} starting with $500? (5/6 when fair)

< 1 / 37,000

What is Pr{reach $1,000,100} starting with $1,000,000? (≈ 1 when fair)

< 1 / 37,000

no matter how many $ at start!



## Gambler's Ruin

- Play $1 bets until going broke or make enough money.

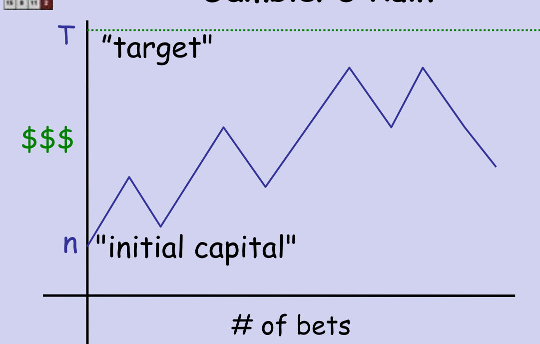
Parameters

- p ::= Pr{win $1 bet}
- n ::= initial capital
- T ::= gambler's target

Question: What is Pr{reach target}?



## Gambler's Ruin



## Dow Jones Trend

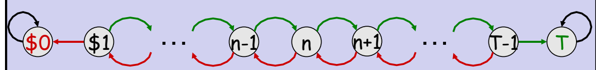

random steps with “up” bias?



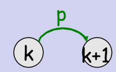


## Gambler’s Ruin

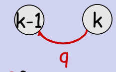
View as random walk on a line.




- p ::=Pr{win a bet}



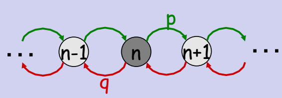

- q ::= 1-p = Pr{lose a bet}




What is Pr{reach T before 0}?



## General Approach



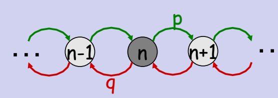

$w_n$ ::= Pr{hit target | start at n}

$= w_{n+1} \cdot p$

$+ w_{n-1} \cdot q$



## General Approach




$w_n = pw_{n+1} + qw_{n-1}$, so

$w_{n+1} = (1/p)w_n - (q/p)w_{n-1}$

**we have a linear recurrence!**



## Linear Recurrence

$w_{n+1} = (1/p)w_n - (q/p)w_{n-1}$

$w_0 = 0$ (Gambler is broke)

$w_T = 1$ (Gambler is at target)

Solve using generating functions and get:



## Linear Recurrence

$$w_n = \frac{r^n - 1}{r - 1} \cdot w_1$$

Twist: we don’t know $w_1$

for $r ::= q/p \neq 1$.

But $w_T = 1$, so can solve for $w_1$. Then

$$w_n = \frac{r^n - 1}{r^T - 1}$$

## Winning when Biased Against

$$w_n = \frac{r^n - 1}{r^T - 1} < \frac{r^n}{r^T} = \left(\frac{1}{r}\right)^{T-n}$$

intended profit ($T-n$)

Suppose $p < q$, so $r ::= q/p > 1$.



## Profit \$100 in US Roulette

$p = 18/38$  $q = 20/38$  $1/r = 9/10$

$$\Pr\{\text{Profit } \$100\} < (9/10)^{100}$$

$< 1/37{,}648$



## Profit \$200 in US Roulette

$p = 18/38$  $q = 20/38$  $1/r = 9/10$

$$\Pr\{\text{Profit } \$200\} < (9/10)^{200}$$

$< 1/70{,}000{,}000$



## What About the Fair Case?

$$w_n = \frac{r^n - 1}{r^T - 1} \qquad (r ::= q/p = 1)$$

- Uh oh, dividing by 0.
- Use l'Hôpital's Rule

$$\lim_{r \to 1} \frac{d(r^n-1)/dr}{d(r^T-1)/dr} = \frac{nr^{n-1}}{Tr^{T-1}} = \frac{n}{T}$$



## Team Problems

# Problems

# 1 & 2