Solutions to In-Class Problems Week 13, Wed.

Problem 1.
Let's see what it takes to make Carnival Dice fair. Here's the game with payoff parameter k: make three independent rolls of a fair die. If you roll a six

- no times, then you lose 1 dollar.
- exactly once, then you win 1 dollar.
- exactly twice, then you win two dollars.
- all three times, then you win k dollars.

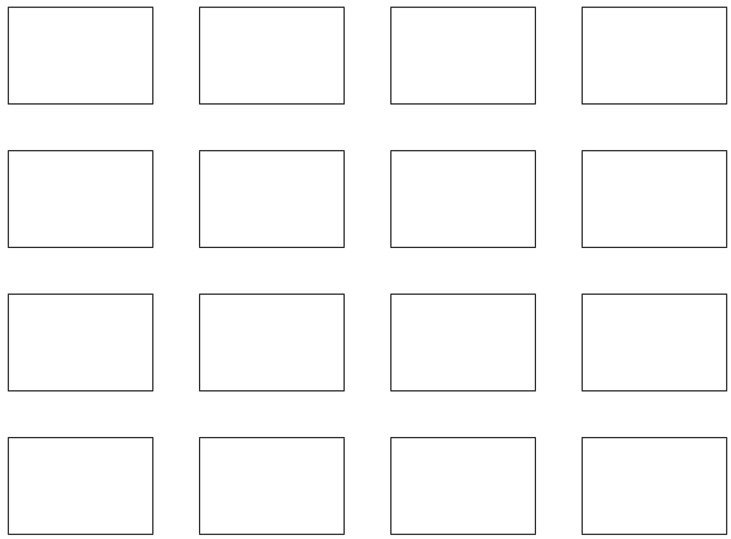
For what value of k is this game fair?

Solution. Let the random variable P be your payoff. Then we can compute $\mathrm{E}[P]$ as follows:

$$\begin{aligned}\mathrm{E}[P] &= -1 \cdot \Pr\{0 \text{ sixes}\} + 1 \cdot \Pr\{1 \text{ six}\} + 2 \cdot \Pr\{2 \text{ sixes}\} + k \cdot \Pr\{3 \text{ sixes}\} \\ &= -1 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + 2 \cdot 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + k \cdot \left(\frac{1}{6}\right)^3 \\ &= \frac{-125 + 75 + 30 + k}{216}\end{aligned}$$

The game is fair when $\mathrm{E}[P] = 0$. This happens when $k = 20$. ■

Problem 2.
A classroom has sixteen desks arranged as shown below.



If there is a girl in front, behind, to the left, or to the right of a boy, then the two of them *flirt*. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied by boys and girls with equal probability and mutually independently. What is the expected number of flirting couples? *Hint:* Linearity.

Solution. First, let's count the number of pairs of adjacent desks. There are three in each row and three in each column. Since there are four rows and four columns, there are $3 \cdot 4 + 3 \cdot 4 = 24$ pairs of adjacent desks.

Number these pairs of adjacent desks from 1 to 24. Let F_i be an indicator for the event that occupants of the desks in the i-th pair are flirting. The probability we want is then:

$$\begin{aligned} \mathrm{E}\left[\sum_{i=1}^{24} F_i\right] &= \sum_{i=1}^{24} \mathrm{E}\left[F_i\right] && \text{(linearity of } \mathrm{E}\left[\cdot\right]\text{)} \\ &= \sum_{i=1}^{24} \Pr\left\{F_i = 1\right\} && (F_i \text{ is an indicator}) \end{aligned}$$

The occupants of adjacent desks are flirting iff they are of opposite sexes, which happens with probability $1/2$, that is, $\Pr\left\{F_i = 1\right\} = 1/2$. Plugging this into the previous expression gives:

$$\mathrm{E}\left[\sum_{i=1}^{24} F_i\right] = \sum_{i=1}^{24} \Pr\left\{F_i = 1\right\} = 24 \cdot \frac{1}{2} = 12$$

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Problem 3. (a) Suppose we flip a fair coin until two Tails in a row come up. What is the expected number, N_{TT}, of flips we perform? *Hint:* Let D be the tree diagram for this process. Explain why $D = H \cdot D + T \cdot (H \cdot D + T)$. Use the Law of Total Expectation 20.3.5

Solution. $N_{\texttt{TT}} = 6$.

From D and Total Expectation:

$$N_{\texttt{TT}} = \frac{1}{2} \cdot [1 + N_{\texttt{TT}}] + \frac{1}{2} \cdot \left(1 + \frac{1}{2} \cdot [1 + N_{\texttt{TT}}] + \frac{1}{2} \cdot 1\right)$$

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(b) Suppose we flip a fair coin until a Tail immediately followed by a Head come up. What is the expected number, $N_{\texttt{TH}}$, of flips we perform?

Solution. $N_{\texttt{TH}} = 4$.

This time the tree diagram $C = H \cdot C + T \cdot B$ where the subtree $B = H + T \cdot B$.

So

$$N_{\texttt{TH}} = \frac{1}{2} \cdot [1 + N_{\texttt{TH}}] + \frac{1}{2} \cdot [1 + N_B]$$

where N_B is the expected number of flips in the B subtree. But

$$N_B = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot [1 + N_B].$$

That is, $N_B = 2$. Hence,

$$N_{\texttt{TH}} = \frac{1}{2} \cdot [1 + 2] + \frac{1}{2} \cdot [1 + N_{\texttt{TH}}]$$

which implies $N_{\texttt{TH}} = 4$. ■

(c) Suppose we now play a game: flip a fair coin until either `TT` or `TH` first occurs. You win if `TT` comes up first, lose if `TH` comes up first. Since `TT` takes 50% longer on average to turn up, your opponent agrees that he has the advantage. So you tell him you're willing to play if you pay him $5 when he wins, but he merely pays you a 20% premium, that is, $6, when you win.

If you do this, you're sneakily taking advantage of your opponent's untrained intuition, since you've gotten him to agree to unfair odds. What is your expected profit per game?

Solution. It's easy to see that both `TT` and `TH` are equally likely to show up first. (Every game play consists of a sequence of `H`'s followed by a `T`, after which the game ends with a `T` or an `H`, with equal probability.) So your expected profit is

$$\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot (-5)$$

dollars, that is 50 cents per game. So leap to play. ■

Problem 4.

Justify each line of the following proof that if R_1 and R_2 are *independent*, then

$$\mathrm{E}[R_1 \cdot R_2] = \mathrm{E}[R_1] \cdot \mathrm{E}[R_2].$$

Proof.

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\begin{aligned}
&\mathrm{E}[R_1 \cdot R_2] \\
&= \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr\{R_1 \cdot R_2 = r\} \\
&= \sum_{r_i \in \text{range}(R_i)} r_1 r_2 \cdot \Pr\{R_1 = r_1 \text{ and } R_2 = r_2\} \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr\{R_1 = r_1 \text{ and } R_2 = r_2\} \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr\{R_1 = r_1\} \cdot \Pr\{R_2 = r_2\} \\
&= \sum_{r_1 \in \text{range}(R_1)} \left(r_1 \Pr\{R_1 = r_1\} \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr\{R_2 = r_2\} \right) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr\{R_1 = r_1\} \cdot \mathrm{E}[R_2] \\
&= \mathrm{E}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr\{R_1 = r_1\} \\
&= \mathrm{E}[R_2] \cdot \mathrm{E}[R_1].
\end{aligned}
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Solution. *Proof.*

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\begin{aligned}
&\mathrm{E}[R_1 \cdot R_2] \\
&::= \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr\{R_1 \cdot R_2 = r\} && \text{(by definition)} \\
&= \sum_{r_i \in \text{range}(R_i)} r_1 r_2 \cdot \Pr\{R_1 = r_1 \text{ AND } R_2 = r_2\} && \text{(event } [R_1 \cdot R_2 = r] \text{ splits into events} \\
& && [R_1 = r_1 \text{ AND } R_2 = r_2] \text{ such that } r_1 r_2 = r) \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr\{R_1 = r_1 \text{ AND } R_2 = r_2\} && \text{(ordering terms in the sum)} \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr\{R_1 = r_1\} \cdot \Pr\{R_2 = r_2\} && \text{(independence of } R_1, R_2) \\
&= \sum_{r_1 \in \text{range}(R_1)} \left(r_1 \Pr\{R_1 = r_1\} \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr\{R_2 = r_2\} \right) && \text{(factor out } r_1 \Pr\{R_1 = r_1\}) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr\{R_1 = r_1\} \cdot \mathrm{E}[R_2] && \text{(def of } \mathrm{E}[R_2]) \\
&= \mathrm{E}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr\{R_1 = r_1\} && \text{(factor out } \mathrm{E}[R_2]) \\
&= \mathrm{E}[R_2] \cdot \mathrm{E}[R_1]. && \text{(def of } \mathrm{E}[R_1])
\end{aligned}
$$

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