Mathematics for Computer Science
MIT 6.042J/18.062J

Great Expectations

Carnival Dice

choose a number from 1 to 6,
then roll 3 fair dice:
win $1 for each match
lose $1 if no match

Carnival Dice

Example: choose 5, then
roll 2,3,4: lose $1
roll 5,4,6: win $1
roll 5,4,5: win $2
roll 5,5,5: win $3

Carnival Dice

Is this a fair game?

Carnival Dice

# matches	probability	$ won
0	125/216	-1
1	75/216	1
2	15/216	2
3	1/216	3

Carnival Dice

so every 216 games, expect
0 matches about 125 times
1 match about 75 times
2 matches about 15 times
3 matches about once

Carnival Dice

So on average expect to win:

$125 \cdot$ **NOT fair!** $1 \cdot 3$

$= -\frac{17}{216} \approx -8\,\text{cents}$

Carnival Dice

You can "expect" to lose 8 cents per play. But you never actually lose 8 cents on any single play, this is just your average loss.

Expected Value

The expected value of random variable R is the average value of R --with values weighted by their probabilities

Expected Value

The expected value of random variable R is

$E[R] ::= \sum v \cdot \Pr\{R = v\}$

so E[\$win in Carnival] = $-\frac{17}{216}$

Expected Value

Alternative definition:

$$E[R] = \sum_{\omega \in S} R(\omega) \cdot \Pr\{\omega\}$$

both forms are useful

Expected Value

also called

mean value, mean, or expectation

Indicator Variables

The indicator variable for event A:

$$I_A ::= \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } \overline{A} \text{ occurs.} \end{cases}$$

(Sanity check:

I_A and I_B are independent iff A and B are independent.)

Expectation of indicator I

$$E[I] = 1 \cdot \Pr\{I=1\} + 0 \cdot \Pr\{I=0\}$$

$$= \Pr\{I=1\}$$

Expected #Heads

n independent flips of a coin with bias p for Heads. How many Heads expected?

$$E\left[B_{n,p}\right] = \sum_{k=0}^{n} k \cdot \Pr\left\{k \text{ Heads}\right\}$$

$$::= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Expected #Heads

we know how to get a closed formula for this sum, but we'll see simpler approaches soon.

$$E\left[B_{n,p}\right] = \sum_{k=0}^{n} k \cdot \Pr\left\{k \text{ Heads}\right\}$$

$$::= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Law of Total Expectation

conditional expectation:

$$E[R \mid A] ::= \sum v \cdot \Pr\{R = v \mid A\}$$

$$E[R] = E[R \mid A] \cdot \Pr\{A\} + E[R \mid \overline{A}] \cdot \Pr\{\overline{A}\}$$

good for reasoning by cases

Expected #Heads

Let $e(n)$ ::= expected #H's in n flips.

$= 1 + e(n-1)$ if 1st flip H

$= e(n-1)$ if 1st flip T

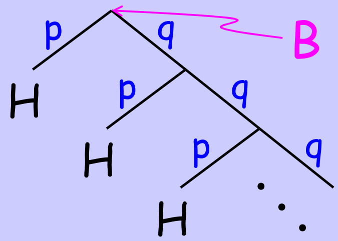
by Total Expectation:

$$e(n) = [1 + e(n-1)] \cdot p + e(n-1) \cdot q$$

$$e(n) = e(n-1) + p = np = E\left[B_{n,p}\right]$$

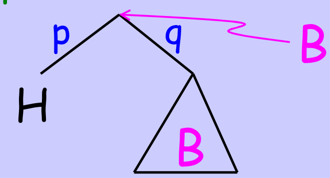
Mean Time to "Failure"

E[# flips until first head]?



Mean Time to "Failure"

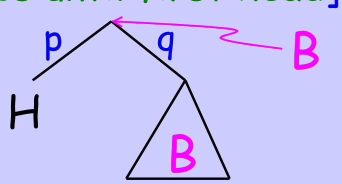
E[# flips until first head]?



now use Total Expectation

Mean Time to "Failure"

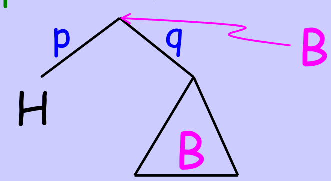
E[# flips until first head]?



$$E = \underbrace{E[\# \mid 1^{st} \text{ is } H]}_{1}\cdot p + \underbrace{E[\# \mid 1^{st} \text{ is } T]}_{1+E}\cdot q$$

Mean Time to "Failure"

E[# flips until first head]?



$$E = 1 \cdot p + [E+1] \cdot (1-p)$$

now solve for E

Mean Time to "Failure"

E[# flips until first head]

$$= \frac{1}{p}$$

Mean Time to Failure

application: if space station Mir has 1/150,000 chance of exploding in any given hour, after how may hours do we expect it to explode?

150,000 hours $\approx$ 17 years

Linearity of Expectation

R,S random variables, a,b constants

$$E[aR + bS] = aE[R] + bE[S]$$

even if R,S are dependent

Expected #Heads

$$\#H's = H_1 + H_2 + \cdots + H_n$$

where H_i is indicator for Head on ith flip

Expected #Heads

$$E[\#H's] = E[H_1 + H_2 + \cdots + H_n]$$

so by linearity

$$= E[H_1] + E[H_2] + \cdots + E[H_n]$$

$$= n \cdot E[H_1] = np$$

Expected #hats returned

Say n people with hats leave their hats at a hat-check station. The hats get totally scrambled randomly. How many hats do we expect will be returned to their owners?

Expected #hats returned

Let R_i be indicator for ith hat being returned to its owner

R_i and R_j are not independent!

Expected #hats returned

Let R_i be indicator for ith hat being returned to its owner

Then E[# hats returned] =

$$E[\Sigma_i R_i] = \Sigma_i E[R_i] = \Sigma_i \Pr\{R_i=1\} = \Sigma_i 1/n = n(1/n) = 1$$

Expectation & Independence

for independent R,S

$$E[R \cdot S] = E[R] \cdot E[S]$$

Team Problems

Problems 1 — 4