Solutions to In-Class Problems Week 13, Mon.

Problem 1.
Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr\{F_i\} = p$$

for $1 \leq i \leq n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr\{F\}$. However, we can give useful upper and lower bounds, namely,

$$p \leq \Pr\{F\} \leq np. \tag{1}$$

We may as well assume $p < 1/n$, since the upper bound is trivial otherwise. For example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \ldots, n\})$ whose outcomes are subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the indices of exactly those components that fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

Solution. There could be a probability p of system failure if all the individual failures occur together. That is, let $\Pr\{\{1, \ldots, n\}\} ::= p$, $\Pr\{\emptyset\} ::= 1 - p$, and let the probability of all other outcomes be zero. So $F_i = \{s \in \mathcal{S} \mid i \in s\}$ and $\Pr\{F_i\} = 0 + 0 + \cdots + 0 + \Pr\{\{1, \ldots, n\}\} = \Pr\{\{1, \ldots, n\}\} = p$. Also, the only outcome with positive probability in F is $\{1, \ldots, n\}$, so $\Pr\{F\} = p$, as required. ■

(b) Show that the probability that the system fails could actually be as large as np by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

Solution. Suppose at most one component ever fails at a time. That is, $\Pr\{\{i\}\} = p$ for $1 \le i \le n$, $\Pr\{\emptyset\} = 1 - np$, and probability of all other outcomes is zero. The sum of the probabilities of all the outcomes is one, so this is a well-defined probability space. Also, the only outcome in F_i with positive probability is $\{i\}$, so $\Pr\{F_i\} = \Pr\{\{i\}\} = p$ as required. Finally, $\Pr\{F\} = np$ because $F = \{A \subseteq \{1, \ldots, n\} \mid A \neq \emptyset\}$, so F in particular contains all the n outcomes of the form $\{i\}$. ∎

(c) Prove inequality (1).

Solution. $F = \bigcup_{i=1}^{n} F_i$ so

$$\begin{aligned}
p &= \Pr\{F_1\} && \text{(given)} && (2)\\
&\le \Pr\{F\} && \text{(since } F_1 \subseteq F\text{)} && (3)\\
&= \Pr\left\{\bigcup F_i\right\} && \text{(def. of } F\text{)} && (4)\\
&\le \sum_{i=1}^{n} \Pr\{F_i\} && \text{(Union Bound)} && (5)\\
&= np. && \text{(since the } F_i\text{'s are disjoint)} && (6)
\end{aligned}$$

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(d) Describe probabilities for the outcomes so that the component failures are mutually independent.

Solution.

$$\Pr\{s\} ::= p^{|s|}(1-p)^{n-|s|}$$

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Guess the Bigger Number Game

Team 1:

- Write different integers between 0 and 7 on two pieces of paper.
- Put the papers face down on a table.

Team 2:

- Turn over one paper and look at the number on it.
- Either stick with this number or switch to the unseen other number.

Team 2 wins if it chooses the larger number.

Problem 2.
In section 20.2.3, Team 2 was shown to have a strategy that wins 4/7 of the time no matter how Team 1 plays. Can Team 2 do better? The answer is "no," because Team 1 has a strategy that guarantees that it wins at least 3/7 of the time, no matter how Team 2 plays. Describe such a strategy for Team 1 and explain why it works.

Solution. Team 1 should randomly choose a number $Z \in \{0, \ldots, 6\}$ and write Z and $Z + 1$ on the pieces of paper with all numbers equally likely.

To see why this works, let N be the number on the paper that Team 2 turns over, and let OK be the event that $N \in \{1, \ldots, 6\}$. So given event OK, that is, given that $N \in \{1, \ldots, 6\}$, Team 1's strategy ensures that half the time N is the higher number and half the time N is the lower number. So given event OK, the probability that Team 1 wins is exactly 1/2 *no matter how Team 2 chooses to play* (stick or switch).

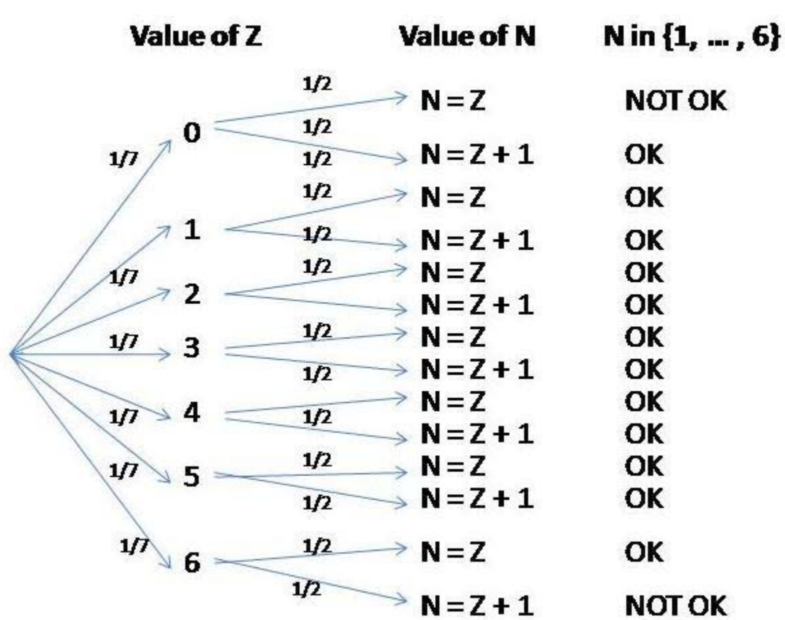
Now we claim that

$$\Pr\{\text{OK}\} = \frac{6}{7}, \tag{7}$$

which implies that the probability that Team 1 wins is indeed at least $(1/2)(6/7) = 3/7$.

To prove $\Pr\{\text{OK}\} = 6/7$, we can follow the four step method. (Note that we couldn't apply this method to model the behavior of Team 2, since we don't know how they may play, and so we can't let our analysis depend on what they do.)

The first level of the probability tree for this game will describe the value of Z: there are seven branches from the root with equal probability going to first level nodes corresponding to the seven possible values of Z. The second level of the tree describes the choice of the number, N: each of the seven first-level nodes has two branches with equal probability, one branch for the case that $N = Z$ and the other for the case that $N = Z + 1$. So there are 14 outcome (leaf) nodes at the second level of the tree, each with probability 1/14.



Now only two outcomes are not OK, namely, when $Z = 6$ and $N = 7$, and when $Z = 0$ and $N = 0$. Each of the other twelve outcomes is OK, and since each has probability $1/14$, we conclude that $\Pr\{\text{OK}\} = 12/14 = 6/7$, as claimed. ■

Problem 3.
Suppose X_1, X_2, and X_3 are three mutually independent random variables, each having the uniform distribution

$$\Pr\{X_i = k\} \text{ equal to } 1/3 \text{ for each of } k = 1, 2, 3.$$

Let M be another random variable giving the maximum of these three random variables. What is the density function of M?

Solution.

$$\begin{aligned}\text{PDF}_M(1) &= \tfrac{1}{27} \\ \text{PDF}_M(2) &= \tfrac{7}{27} \\ \text{PDF}_M(3) &= \tfrac{19}{27}\end{aligned}$$

This can be hashed out by counting the possible outcomes. Alternatively, we can reason as follows:

The event $M = 1$ is the event that all three of the variables equal 1, and since they are mutually independent, we have

$$\Pr\{M = 1\} = \Pr\{X_1 = 1\} \cdot \Pr\{X_2 = 1\} \cdot \Pr\{X_3 = 1\} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

To compute $\Pr\{M = 2\}$, we first compute $\Pr\{M \leq 2\}$. Now the event $[M \leq 2]$ is the event that all three of the variables is at most 2, so by mutual independence we have

$$\Pr\{M \leq 2\} = \Pr\{X_1 \leq 2\} \cdot \Pr\{X_2 \leq 2\} \cdot \Pr\{X_3 \leq 2\} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

Therefore,

$$\Pr\{M = 2\} = \Pr\{M \leq 2\} - \Pr\{M = 1\} = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}.$$

Finally,

$$\Pr\{M = 3\} = 1 - \Pr\{M \leq 2\} = 1 - \frac{8}{27} = \frac{19}{27}.$$

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Problem 4.
Suppose you have a biased coin that has probability p of flipping heads. Let J be the number of heads in n independent coin flips. So J has the general binomial distribution:

$$\mathrm{PDF}_J(k) = \binom{n}{k} p^k q^{n-k}$$

where $q ::= 1 - p$.

(a) Show that

$$\begin{aligned} \mathrm{PDF}_J(k) &< \mathrm{PDF}_J(k+1) && \text{for } k < np + p, \\ \mathrm{PDF}_J(k) &> \mathrm{PDF}_J(k+1) && \text{for } k > np + p. \end{aligned}$$

Solution. Consider the ratio of the probability of k heads over the probability of $k-1$ heads.

$$\begin{aligned} \frac{\mathrm{PDF}_J(k)}{\mathrm{PDF}_J(k-1)} &= \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} \\ &= \frac{\frac{n!}{k!\,(n-k)!}}{\frac{n!}{(k-1)!\,(n-k+1)!}} \frac{p}{q} \\ &= \frac{(n-k+1)p}{kq} \end{aligned}$$

This fraction is greater than 1 precisely when $(n-k+1)p > kq = k(1-p)$, that is when $k < np+p$. So for $k < np + p$, the probability of k heads increases as k increases, and for $k > np + p$, the probability decreases as k increases. ■

(b) Conclude that the maximum value of PDF_J is asymptotically equal to

$$\frac{1}{\sqrt{2\pi npq}}.$$

Hint: For the asymptotic estimate, it's ok to assume that np is an integer, so by part (a) the maximum value is $\mathrm{PDF}_J(np)$. Use Stirling's formula 15.12[1].

[1] $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

Solution.

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\begin{aligned}
\mathrm{PDF}_J(np) &::= \binom{n}{np} p^{np} q^{n-np} \\
&= \frac{n!}{(np)!\,(nq)!} p^{np} q^{nq} \\
&\sim \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{\left(\left(\frac{np}{e}\right)^{np} \sqrt{2\pi np}\right)\left(\left(\frac{nq}{e}\right)^{nq} \sqrt{2\pi nq}\right)} p^{np} q^{nq} \\
&= \frac{\frac{n^n}{e^n}\sqrt{2\pi n}}{\left(\frac{n^{np}p^{np}}{e^{np}}\sqrt{2\pi np}\right)\left(\frac{n^{nq}q^{nq}}{e^{nq}}\sqrt{2\pi nq}\right)} p^{np} q^{nq} \\
&= \frac{\frac{n^n}{e^n}\sqrt{2\pi n}}{\frac{n^{np+nq}p^{np}q^{nq}}{e^{np+nq}}\sqrt{2\pi np}\sqrt{2\pi nq}} p^{np} q^{nq} p^{np} q^{nq} \\
&= \frac{\frac{n^n}{e^n}\sqrt{2\pi n}}{\frac{n^n}{e^n}\sqrt{2\pi np}\sqrt{2\pi nq}} \\
&= \frac{1}{\sqrt{2\pi npq}}.
\end{aligned}
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