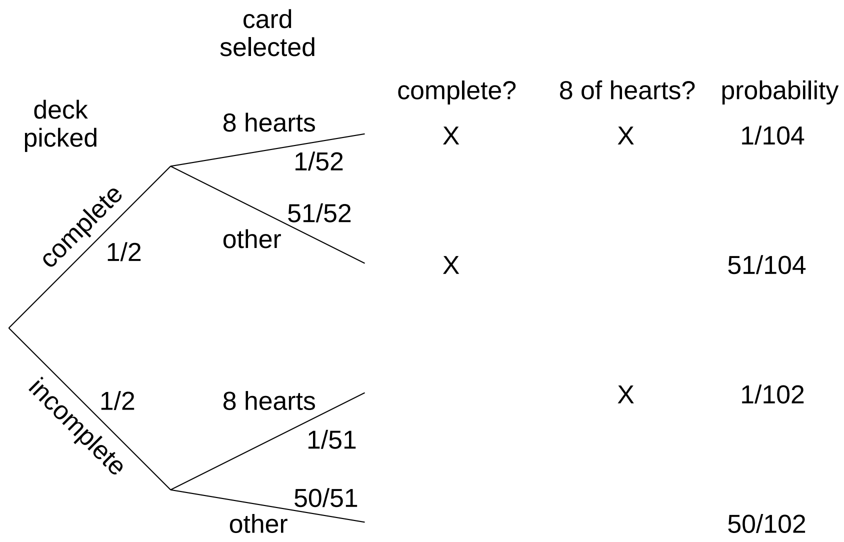
Solutions to In-Class Problems Week 12, Fri.

Problem 1.
There are two decks of cards. One is complete, but the other is missing the ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

Solution. Let C be the event that you pick the complete deck, and let H be the event that you select the eight of hearts. In these terms, our aim is to compute:

$$\Pr\{C \mid H\} = \frac{\Pr\{C \cap H\}}{\Pr\{H\}}$$

A tree diagram is worked out below:



Now we can compute the desired conditional probability as follows:

$$
\begin{aligned}
\Pr\{C \mid H\} &= \frac{\Pr\{C \cap H\}}{\Pr\{H\}} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{52}}{\frac{1}{2} \cdot \frac{1}{52} + \frac{1}{2} \cdot \frac{1}{51}} \\
&= \frac{51}{103} \\
&= 0.495146\ldots
\end{aligned}
$$

Thus, if you selected the eight of hearts, then the deck you picked is less likely to be the complete one. It's worth thinking about how you might have arrived at this final conclusion without going through the detailed calculation.

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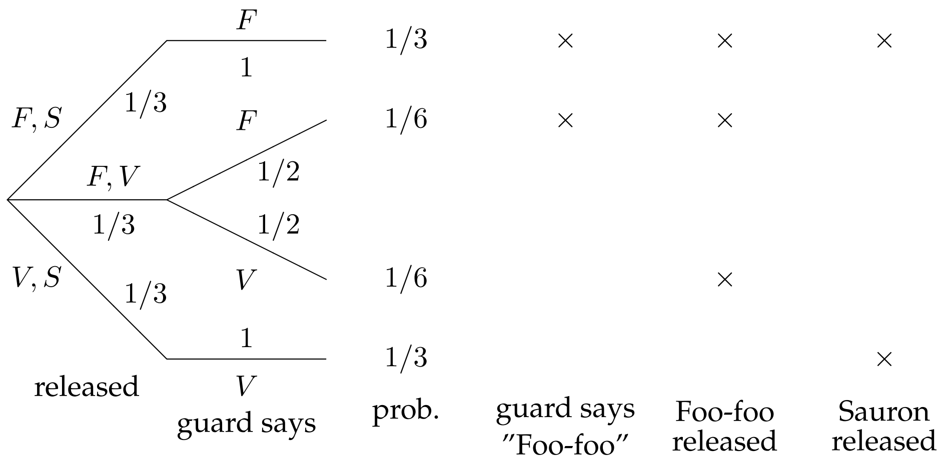
Problem 2.
There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $2/3$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that if the guard says, for example, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $1/2$. This is because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Using a tree diagram and the four-step method, either prove that the Dark Lord Sauron has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), then he names one of the two uniformly at random.

Solution. Sauron has reasoned incorrectly. In order to understand his error, let's begin by working out the sample space, noting events of interest, and computing outcome probabilities:



Define the events S, F, and "F" as follows:

$$\begin{aligned} \text{“}F\text{”} &= \text{Guard says Foo-Foo is released} \\ F &= \text{Foo-Foo is released} \\ S &= \text{Sauron is released} \end{aligned}$$

The outcomes in each of these events are noted in the tree diagram.

Sauron's error is in failing to realize that the event F (Foo-foo will be released) is different from the event "F" (the guard *says* Foo-foo will be released). In particular, the probability that Sauron is released, given that Foo-foo is released, is indeed $1/2$:

$$\begin{aligned} \Pr\{S \mid F\} &= \frac{\Pr\{S \cap F\}}{\Pr\{F\}} \\ &= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6}} \\ &= \frac{1}{2} \end{aligned}$$

But the probability that Sauron is released given that the guard merely *says so* is still $2/3$:

$$\begin{aligned} \Pr\{S \mid \text{“}F\text{”}\} &= \frac{\Pr\{S \cap \text{“}F\text{”}\}}{\Pr\{\text{“}F\text{”}\}} \\ &= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} \\ &= \frac{2}{3} \end{aligned}$$

So Sauron's probability of release is actually unchanged by the guard's statement. ■

Problem 3.
Suppose that you flip three fair, mutually independent coins. Define the following events:

- Let A be the event that *the first* coin is heads.

- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

(a) Use the four step method to determine the probability space for this experiment and the probability of each of A, B, C, D.

Solution. The tree is a binary tree with depth 3 and 8 leaves. The successive levels branching to show whether or not the successive events A, B, C occur. By definition of *fair* and *independent*, each branch out of a vertex is equally likely to be followed. So the probability space has as outcomes the eight length-3 strings of `H`'s and `T`'s, each of which has probability $(1/2)^3 = 1/8$.

Each of the events events A, B, C, D are true in four of the outcomes and hence has probability $1/2$. ■

(b) Show that these events are not mutually independent.

Solution.

$$\Pr\{A \cap B \cap C \cap D\} = 0 \neq (1/2)^4 = \Pr\{A\} \cdot \Pr\{B\} \cdot \Pr\{C\} \cdot \Pr\{D\}.$$

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(c) Show that they are 3-way independent.

Solution. Because the coin tosses are mutually independent, we know:

$$\Pr\{A \cap B \cap C\} = \Pr\{A\} \cdot \Pr\{B\} \cdot \Pr\{C\}.$$

What remains is to check that equality holds for the other subsets of three events: $\{A, B, D\}$, $\{A, C, D\}$, and $\{B, C, D\}$. By symmetry, again, we need only check one, say the first one.

$$\Pr\{A \cap B \cap D\} = \Pr\{\{HHT\}\} = \frac{1}{8}.$$

Since this is equal to $\Pr\{A\} \cdot \Pr\{B\} \cdot \Pr\{D\}$, these three events are independent.

We conclude that all four events are three-way independent. ■