In-Class Problems Week 12, Fri.

Problem 1.
There are two decks of cards. One is complete, but the other is missing the ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

Problem 2.
There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $2/3$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that if the guard says, for example, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $1/2$. This is because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Using a tree diagram and the four-step method, either prove that the Dark Lord Sauron has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), then he names one of the two uniformly at random.

Problem 3.
Suppose that you flip three fair, mutually independent coins. Define the following events:

- Let A be the event that *the first* coin is heads.
- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

(a) Use the four step method to determine the probability space for this experiment and the probability of each of A, B, C, D.

(b) Show that these events are not mutually independent.

(c) Show that they are 3-way independent.

Appendix

The Four Step Method

This is a good approach to questions of the form, "What is the probability that ——–?" Intuition can be misleading, but this formal approach gives the right answer every time.

1. Find the sample space. (Use a tree diagram.)
2. Define events of interest. (Mark leaves corresponding to these events.)
3. Determine outcome probabilities:
   (a) Assign edge probabilities.
   (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)
4. Compute event probabilities. (Sum the probabilities of all outcomes in the event.)

Conditional Probabilitiy

For events E, F such that $\Pr\{F\} \neq 0$, the *conditional probability* of E given F is:

$$\Pr\{E \mid F\} ::= \frac{\Pr\{E \cap F\}}{\Pr\{F\}}$$

Law of Total Probability

Here is the Law stated for three sets: suppose E, F, G are pairwise disjoint events, and

$$A \subseteq E \cup F \cup G.$$

Then

$$\begin{aligned}\Pr\{A\} &= \Pr\{A \cap E\} + \Pr\{A \cap F\} + \Pr\{A \cap G\} \\ &= \Pr\{A \mid E\} \cdot \Pr\{E\} + \Pr\{A \mid F\} \cdot \Pr\{F\} + \Pr\{A \mid G\} \cdot \Pr\{G\}.\end{aligned}$$

Independence

Events E, F are *independent* iff

$$\Pr\{E \cap F\} = \Pr\{E\} \cdot \Pr\{F\}.$$

Events $E_1, E_2, \dots, E_n$ are *mutually independent* if and only if

$$\Pr\left\{\bigcap_{i \in J} E_i\right\} = \prod_{i \in J} \Pr\{E_i\}$$

for all subsets $J \subseteq \{1, \dots, n\}$.

Events $E_1, E_2, \dots,$ are k-way independent iff every k of these events are mutually independent.