Solutions to In-Class Problems Week 12, Wed.

Problem 1.
[A Baseball Series]
The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. (In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends.) Assume that the Red Sox win each game with probability $3/5$, regardless of the outcomes of previous games.

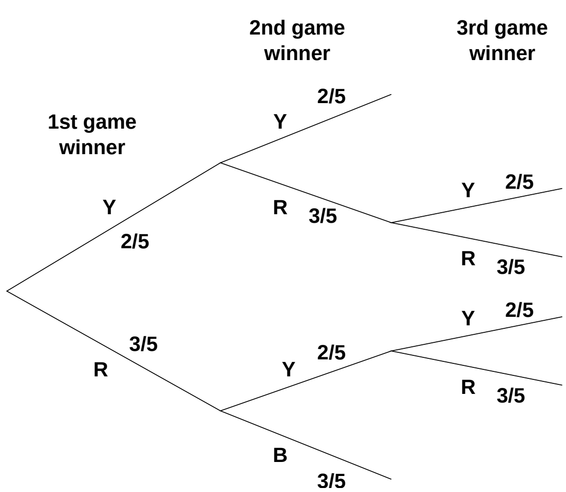
Answer the questions below using the four step method. You can use the same tree diagram for all three problems.

(a) What is the probability that a total of 3 games are played?

(b) What is the probability that the winner of the series loses the first game?

(c) What is the probability that the *correct* team wins the series?

Solution. A tree diagram is worked out below.



outcome	3 games played?	winner lost first game?	correct team wins?	Probability
YY				4/25
YRY	X			12/125
YRR	X	X	X	18/125
RYY	X	X		12/125
RYR	X		X	18/125
RR			X	9/25

From the tree diagram, we get:

$$\Pr\{3 \text{ games played}\} = \frac{12}{125} + \frac{18}{125} + \frac{12}{125} + \frac{18}{125} = \frac{12}{25}$$

$$\Pr\{\text{winner lost first game}\} = \frac{18}{125} + \frac{12}{125} = \frac{6}{25}$$

$$\Pr\{\text{correct team wins}\} = \frac{18}{125} + \frac{18}{125} + \frac{9}{25} = \frac{81}{125}$$

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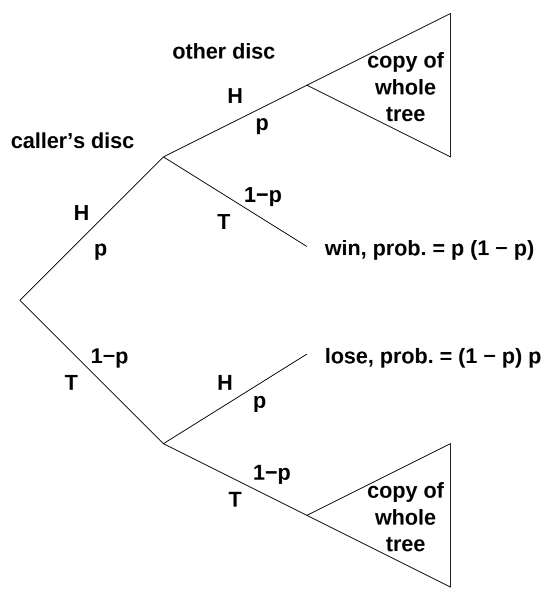
Problem 2.
To determine which of two people gets a prize, a coin is flipped twice. If the flips are a Head and then a Tail, the first player wins. If the flips are a Tail and then a Head, the second player wins. However, if both coins land the same way, the flips don't count and whole the process starts over.

Assume that on each flip, a Head comes up with probability p, regardless of what happened on other flips. Use the four step method to find a simple formula for the probability that the first player wins. What is the probability that neither player wins?

Suggestions: The tree diagram and sample space are infinite, so you're not going to finish drawing the tree. Try drawing only enough to see a pattern. Summing all the winning outcome probabilities directly is difficult. However, a neat trick solves this problem and many others. Let s be the sum of all winning outcome probabilities in the whole tree. Notice that *you can write the sum of all the winning probabilities in certain subtrees as a function of* s. Use this observation to write an equation in s and then solve.

Solution. In the tree diagram below, the small triangles represent subtrees that are themselves complete copies of the whole tree.



Let s equal the sum of all winning probabilities in the whole tree. There are two extra edges with probability p on the path to each outcome in the top subtree. Therefore, the sum of winning probabilities in the upper tree is $p^2 s$. Similarly, the sum of winning probabilities in the lower subtree is $(1-p)^2 s$. This gives the equation:

$$s = p^2 s + (1-p)^2 s + p(1-p)$$

The solution to this equation is $s = 1/2$, for all p between 0 and 1.

By symmetry, the probability that the first player loses is 1/2. This means that the event, if any, of flipping forever can only have probability zero.

Formally, the sample space is the (infinite) set of leaves of the tree, namely,

$$\mathcal{S} ::= \{\texttt{TT}, \texttt{HH}\}^* \cdot \{\texttt{HT}, \texttt{TH}\}$$

where $\{\texttt{TT}, \texttt{HH}\}^*$ denotes the set of strings formed by concatenating a sequence of HH's and TT's. For example,

$$\texttt{TTTTHHHT}, \texttt{HHTTTH}, \texttt{HHHHHHHHHT}, \texttt{HT} \in \mathcal{S}.$$

For any string $s \in \mathcal{S}$,

$$\Pr\{s\} ::= p^{\#\texttt{H}\text{'s in } s}(1-p)^{\#\texttt{T}\text{'s in } s}.$$

To verify that is defines a probability space, we must show that $\sum_{s \in \mathcal{S}} \Pr\{s\} = 1$:

$$\begin{aligned}
\sum_{s \in \mathcal{S}} \Pr\{s\} &= \sum_{n \geq 0} \sum_{s \in \mathcal{S}, |s| = 2n+2} p^{\#\texttt{H}\text{'s in } s}(1-p)^{\#\texttt{T}\text{'s in } s} \\
&= \sum_{n \geq 0} \sum_{i+j=n} p^{2i}(1-p)^{2j} p(1-p) && \text{(strings that end in HT)} \\
&\quad + \sum_{n \geq 0} \sum_{i+j=n} p^{2i}(1-p)^{2j} p(1-p) && \text{(strings that end in TH)} \\
&= 2p(1-p) \sum_{n \in \mathbb{N}} \left(p^2 + (1-p)^2\right)^n \\
&= \frac{2p(1-p)}{1 - (p^2 + (1-p)^2)} \\
&= \frac{2p(1-p)}{2p^2 + 2p} = 1.
\end{aligned}$$

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Problem 3.
Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr\{F_i\} = p$$

for $1 \leq i \leq n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr\{F\}$. However, we can give useful upper and lower bounds, namely,

$$p \leq \Pr\{F\} \leq np. \tag{1}$$

We may as well assume $p < 1/n$, since the upper bound is trivial otherwise. For example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \ldots, n\})$ whose outcomes are subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the indices of exactly those components that fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

Solution. There could be a probability p of system failure if all the individual failures occur together. That is, let $\Pr\{\{1, \ldots, n\}\} ::= p$, $\Pr\{\emptyset\} ::= 1 - p$, and let the probability of all other outcomes be zero. So $F_i = \{s \in \mathcal{S} \mid i \in s\}$ and $\Pr\{F_i\} = 0 + 0 + \cdots + 0 + \Pr\{\{1, \ldots, n\}\} = \Pr\{\{1, \ldots, n\}\} = p$. Also, the only outcome with positive probability in F is $\{1, \ldots, n\}$, so $\Pr\{F\} = p$, as required. ■

(b) Show that the probability that the system fails could actually be as large as np by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

Solution. Suppose at most one component ever fails at a time. That is, $\Pr\{\{i\}\} = p$ for $1 \leq i \leq n$, $\Pr\{\emptyset\} = 1 - np$, and probability of all other outcomes is zero. The sum of the probabilities of all the outcomes is one, so this is a well-defined probability space. Also, the only outcome in F_i with positive probability is $\{i\}$, so $\Pr\{F_i\} = \Pr\{\{i\}\} = p$ as required. Finally, $\Pr\{F\} = np$ because $F = \{A \subseteq \{1, \ldots, n\} \mid A \neq \emptyset\}$, so F in particular contains all the n outcomes of the form $\{i\}$. ■

(c) Prove inequality (1). You may assume the Union Bound in the Appendix.

Solution. $F = \bigcup_{i=1}^{n} F_i$ so

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\begin{aligned}
p &= \Pr\{F_1\} && \text{(given)} && (2)\\
&\leq \Pr\{F\} && \text{(since } F_1 \subseteq F\text{)} && (3)\\
&= \Pr\left\{\bigcup F_i\right\} && \text{(def. of } F\text{)} && (4)\\
&\leq \sum_{i=1}^{n} \Pr\{F_i\} && \text{(Union Bound)} && (5)\\
&= np. && \text{(since the } F_i\text{'s are disjoint)} && (6)
\end{aligned}
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Problem 4.
Here are some handy rules for reasoning about probabilities that all follow directly from the Disjoint Sum Rule in the Appendix. Prove them.

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\} \qquad \text{(Difference Rule)}$$

Solution. Any set A is the disjoint union of $A - B$ and $A \cap B$, so

$$\Pr\{A\} = \Pr\{A - B\} + \Pr\{A \cap B\}$$

by the Disjoint Sum Rule. ■

$$\Pr\{\overline{A}\} = 1 - \Pr\{A\} \qquad \text{(Complement Rule)}$$

Solution. $\overline{A} ::= \mathcal{S} - A$, so by the Difference Rule

$$\Pr\{\overline{A}\} = \Pr\{\mathcal{S}\} - \Pr\{A\} = 1 - \Pr\{A\}.$$

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$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \qquad \text{(Inclusion-Exclusion)}$$

Solution. $A \cup B$ is the disjoint union of A and $B - A$ so

$$\begin{aligned}
\Pr\{A \cup B\} &= \Pr\{A\} + \Pr\{B - A\} && \text{(Disjoint Sum Rule)} \\
&= \Pr\{A\} + (\Pr\{B\} - \Pr\{A \cap B\}) && \text{(Difference Rule)}
\end{aligned}$$

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$$\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}. \qquad \text{(2-event Union Bound)}$$

Solution. This follows immediately from Inclusion-Exclusion and the fact that $\Pr\{A \cap B\} \geq 0$. ■

$$\text{If } A \subseteq B, \text{ then } \Pr\{A\} \leq \Pr\{B\}. \qquad \text{(Monotonicity)}$$

Solution.

$$\begin{aligned}
\Pr\{A\} &= \Pr\{B\} - (\Pr\{B\} - \Pr\{A\}) \\
&= \Pr\{B\} - (\Pr\{B\} - \Pr\{A \cap B\}) && (\text{since } A = A \cap B) \\
&= \Pr\{B\} - \Pr\{B - A\} && \text{(difference rule)} \\
&\leq \Pr\{B\} && (\text{since } \Pr\{B - A\} \geq 0).
\end{aligned}$$

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