

In-Class Problems Week 12, Wed.

Problem 1.

[A Baseball Series]

The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. (In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends.) Assume that the Red Sox win each game with probability $3/5$, regardless of the outcomes of previous games.

Answer the questions below using the four step method. You can use the same tree diagram for all three problems.

- (a) What is the probability that a total of 3 games are played?
- (b) What is the probability that the winner of the series loses the first game?
- (c) What is the probability that the *correct* team wins the series?

Problem 2.

To determine which of two people gets a prize, a coin is flipped twice. If the flips are a Head and then a Tail, the first player wins. If the flips are a Tail and then a Head, the second player wins. However, if both coins land the same way, the flips don't count and whole the process starts over.

Assume that on each flip, a Head comes up with probability p , regardless of what happened on other flips. Use the four step method to find a simple formula for the probability that the first player wins. What is the probability that neither player wins?

Suggestions: The tree diagram and sample space are infinite, so you're not going to finish drawing the tree. Try drawing only enough to see a pattern. Summing all the winning outcome probabilities directly is difficult. However, a neat trick solves this problem and many others. Let s be the sum of all winning outcome probabilities in the whole tree. Notice that *you can write the sum of all the winning probabilities in certain subtrees as a function of s* . Use this observation to write an equation in s and then solve.

Problem 3.

Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p . That is, letting F_i be the event that the i th component fails within one year, we have

$$\Pr \{F_i\} = p$$

for $1 \leq i \leq n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr\{F\}$. However, we can give useful upper and lower bounds, namely,

$$p \leq \Pr\{F\} \leq np. \quad (1)$$

We may as well assume $p < 1/n$, since the upper bound is trivial otherwise. For example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \dots, n\})$ whose outcomes are subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the indices of exactly those components that fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, \emptyset .

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

(b) Show that the probability that the system fails could actually be as large as np by describing appropriate probabilities for the outcomes. Make sure to verify that the sum of your outcome probabilities is 1.

(c) Prove inequality (1). You may assume the Union Bound in the Appendix.

Problem 4.

Here are some handy rules for reasoning about probabilities that all follow directly from the Disjoint Sum Rule in the Appendix. Prove them.

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\} \quad (\text{Difference Rule})$$

$$\Pr\{\overline{A}\} = 1 - \Pr\{A\} \quad (\text{Complement Rule})$$

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \quad (\text{Inclusion-Exclusion})$$

$$\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}. \quad (\text{2-event Union Bound})$$

$$\text{If } A \subseteq B, \text{ then } \Pr\{A\} \leq \Pr\{B\}. \quad (\text{Monotonicity})$$

Appendix

The Four Step Method

This is a good approach to questions of the form, “What is the probability that ——?” Intuition can be misleading, but this formal approach gives the right answer every time.

1. Find the sample space. (Use a tree diagram.)
2. Define events of interest. (Mark leaves corresponding to these events.)
3. Determine outcome probabilities:
 - (a) Assign edge probabilities.
 - (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)
4. Compute event probabilities. (Sum the probabilities of all outcomes in the event.)

Probability Spaces

A countable *sample space*, \mathcal{S} , is a nonempty countable set. An element $w \in \mathcal{S}$ is called an *outcome*. A subset of \mathcal{S} is called an *event*.

A *probability space* consists of a sample space, \mathcal{S} , and a function $\Pr\{\cdot\} : \mathcal{S} \rightarrow [0, 1]$, called the *probability function*, such that

$$\sum_{w \in \mathcal{S}} \Pr\{w\} = 1.$$

For any event, $E \subseteq \mathcal{S}$, the *probability of E* is defined to be the sum of the probabilities of the outcomes in E :

$$\Pr\{E\} ::= \sum_{w \in E} \Pr\{w\}.$$

Sum Rule & Union Bound

Let E_0, E_1, \dots be a (possibly infinite) sequence of events. These events are said to be *pairwise disjoint* if $E_i \cap E_j = \emptyset$ whenever $i \neq j$.

If these events are pairwise disjoint, then

$$\Pr\left\{\bigcup_{n \geq 0} E_n\right\} = \sum_{n \geq 0} \Pr\{E_n\}. \quad (\text{Disjoint Sum Rule})$$

Even if they are not pairwise disjoint,

$$\Pr\left\{\bigcup_{n \geq 0} E_n\right\} \leq \sum_{i \geq n} \Pr\{E_n\}. \quad (\text{Union Bound})$$

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6.042J / 18.062J Mathematics for Computer Science
Spring 2010

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