Mathematics for Computer Science
MIT 6.042J/18.062J

Introduction to Probability Theory

Counting in Probability

What is the *probability* of getting

exactly two jacks

in a poker hand?

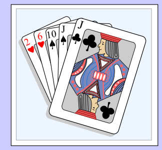
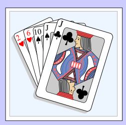


Image by MIT OpenCourseWare.

Counting in Probability

Outcomes: $\binom{52}{5}$ 5-card hands



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Event: $\binom{4}{2}\cdot\binom{52-4}{3}$ hands w/2Jacks

$$\Pr\{2 \text{ Jacks}\} ::= \frac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}} \approx 0.04$$

Probability: 1st Idea

- A set of basic experimental outcomes
- A subset of outcomes is an event
- The probability of an event:

$$\Pr\{\text{event}\} ::= \frac{\#\text{outcomes in event}}{\text{total }\#\text{outcomes}}$$

The Monty Hall Game

Applied Probability:

Let's Make A Deal

(1970's TV Game Show)

Monty Hall Webpages

Image of three doors removed due to copyright restrictions.

http://www.letsmakeadeal.com

The Monty Hall Game

- goats behind two doors
- prize behind third door
- contestant picks a door
- Monty reveals a goat behind an unpicked door
- Contest sticks, or switches to the other unopened door

Analyzing Monty Hall

Marilyn Vos Savant explained Game in magazine -- bombarded by letters (even from PhD's) debating:

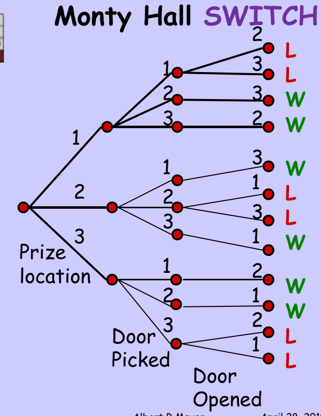
1) sticking & switching equally good
2) switching better

Analyzing Monty Hall

Determine the outcomes.

-- using a tree of possible steps can help

Monty Hall SWITCH strategy



SWITCH
Wins: 6

Lose: 6

Analyzing Monty Hall

A false conclusion:
sticking and switching have same # winning outcomes, so probability of winning is the same for both: 1/2.

Analyzing Monty Hall

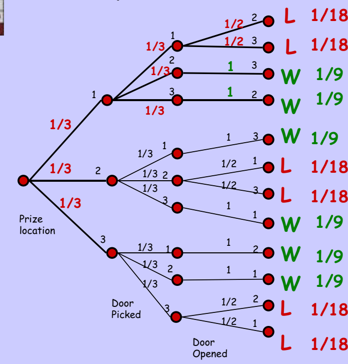
Another false argument:
after door opening, 1 goat and 1 prize are left. Each door is equally likely to have the prize (by symmetry), so

both strategies win with probability: 1/2.

Analyzing Monty Hall

What's wrong?
Let's look at the outcome tree more carefully.

Monty Hall SWITCH strategy



W: 6/9 = **2/3**

L: 6/18 = **1/3**

Probability: 2nd Idea

Outcomes may have differing probabilities!
Not always uniform.

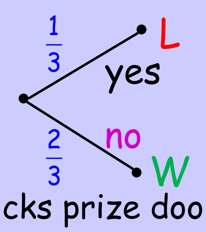
Finding Probability

Intuition is important but dangerous.
Stick with 4-part method:

1. Identify outcomes (*tree helps*)
2. Identify event (*winning*)
3. Assign outcome probabilities
4. Compute event probabilities

really simple analysis

SWITCH strategy wins <u>iff</u> prize door not picked:



picks prize door

$$\Pr\{\text{switch wins}\} = \frac{2}{3}$$

Probability Spaces

1) Sample space: a countable set, $\mathcal{S}$, whose elements are called outcomes.
2) Probability function,

$\Pr: \mathcal{S} \to [0, 1]$, such that

$$\sum_{\omega \in \mathcal{S}} \Pr\{\omega\} = 1.$$

Probability Spaces

An event is a subset, $E \subseteq S$.

$$\Pr\{E\} ::= \sum_{\omega \in E} \Pr\{\omega\}$$

Cor: The Sum Rule

Albert R Meyer, April 28, 2010 lec 12W.22

Sum Rule

For *pairwise disjoint* $A_0, A_1, \ldots$

$$\Pr\{A_0 \cup A_1 \cup A_2 \cup \cdots\} = \Pr\{A_0\} + \Pr\{A_1\} + \Pr\{A_2\} + \cdots$$

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Difference Rule

$$\Pr\{A-B\} = \Pr\{A\} - \Pr\{A \cap B\}$$

because by Sum Rule:

$$\Pr\{A\} = \Pr\{A \cap B\} + \Pr\{A-B\}$$

Albert R Meyer, April 28, 2010 lec 12W.26

Inclusion-Exclusion

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$$

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The Union Bound

$$\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}$$

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Boole's Inequality

for sets $A_0, A_1, \ldots$

$$\Pr\left\{\bigcup_{i \geq 0} A_i\right\} \leq \sum_{i \geq 0} \Pr\{A_i\}$$

Albert R Meyer, April 28, 2010 lec 12W.31

Team Problems

Problems

1—4

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