Mathematics for Computer Science
MIT 6.042J/18.062J

Generating Functions for Recurrences

The Rabbit Population

- A *breeding pair* of rabbits produces a newborn pair every month.
- Rabbits breed after one month.
- After n months:
  w_n::= # newborn pairs
  b_n::= # breeding pairs
- Start with a newborn pair: $w_0 = 1$
  $b_0 = 0$

The Rabbit Population

w_n::= # newborn pairs
b_n::= # breeding pairs

$$b_n = b_{n-1} + w_{n-1} \quad \text{(and so } b_1 = 1\text{)}$$
$$w_n = b_{n-1}$$

Therefore,
$$b_n = b_{n-1} + b_{n-2}$$

The Rabbit Population

$$b_n = b_{n-1} + b_{n-2}$$

It was Fibonacci who was studying rabbit population growth!

Generating Function for Rabbits

$$\begin{aligned}
B(x) &::= b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots \\
-xB(x) &= \quad\;\; - b_0x - b_1x^2 - b_2x^3 - \cdots \\
-x^2B(x) &= \qquad\qquad\;\; - b_0x^2 - b_1x^3 - \cdots \\
& \qquad\qquad\qquad\;\; 0x^2 + 0x^3 + \cdots
\end{aligned}$$

$$b_n - b_{n-1} - b_{n-2} = 0$$

Generating Function for Rabbits

$$\begin{aligned}
B(x) &= b_0 + b_1x \\
-xB(x) & \qquad\; - b_0x \\
-x^2B(x) &
\end{aligned}
\qquad
\begin{pmatrix} b_0 = 0 \\ b_1 = 1 \end{pmatrix}$$

$$B(x) - xB(x) - x^2B(x) = b_0 + (b_1 - b_0)x$$

$$B(x)(1 - x - x^2) = 0 + (1 - 0)x = x$$

Generating Function for Rabbits

$$B(x) = \frac{x}{1 - x - x^2}$$

Coefficient notation

$[x^n]B(x) ::= b_n$ = coeff of x^n in power series for $B(x)$

$$[x^n]\frac{1}{(1 - \alpha x)} = \alpha^n$$

Generating Function for Rabbits

$$B(x) = \frac{x}{1 - x - x^2}$$

use partial fraction expansion to find closed form for $[x^n]B(x)$.

Closed Form for $[x^n]B(x)$

$$B(x) = \frac{x}{1 - x - x^2}$$

factor the denominator

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$$

use quadratic formula

Closed Form for $[x^n]B(x)$

$$B(x) = \frac{x}{1 - x - x^2}$$

factor the denominator

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$$

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}$$

Closed Form for $[x^n]B(x)$

$$B(x) = \frac{a}{1 - \alpha x} + \frac{b}{1 - \beta x}$$

$$\left[x^n\right]B(x) = a \cdot \alpha^n + b \cdot \beta^n$$

need to solve for a and b

Closed Form for $[x^n]B(x)$

$$\frac{a}{1-\alpha x}+\frac{b}{1-\beta x}=\frac{x}{(1-\alpha x)(1-\beta x)}$$

Multiply both sides by $(1-\alpha x)(1-\beta x)$

$$a(1-\beta x)+b(1-\alpha x)=x$$

Solve for a and b —letting x be $1/\alpha$, then $1/\beta$ makes it easy

the answer

$$[x^n]B(x)=$$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

amazing!

the answer

$$[x^n]B(x)=$$

$$\frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\approx 1.62}{}^n-\frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\approx -0.62}{}^n$$

the answer

$$[x^n]B(x)=$$

$$\frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\approx 1.62}{}^n-\frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{\approx (-0.62)^n}$$

converges to 0 as n grows

the answer

$$[x^n]B(x)\sim\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$\approx 0.45(1.62)^n$$

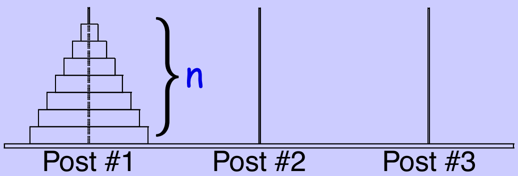
rabbit population grows exponentially

Simpler Closed Form

$$[x^n]B(x)=\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right]$$

round to nearest integer

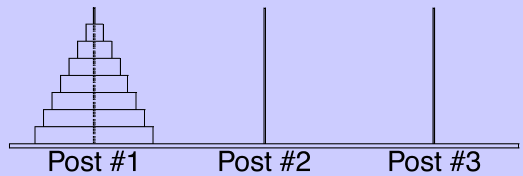
Towers of Hanoi



move stack to Post #2

larger disc not above smaller

Towers of Hanoi



$\text{Move}_{1,2}(n) ::= [\text{Move}_{1,3}(n\text{-}1);$
$\quad \text{big disk } 1 \rightarrow 2;$
$\quad \text{Move}_{3,2}(n\text{-}1)]$

Towers of Hanoi

h_n::=# steps by $\text{Move}_{1,2}(n)$

$$h_n = 2h_{n-1} + 1$$

$$h_0 = 0$$

Hanoi Generating Function

$h_n = 2h_{n-1} + 1$

$h_0 = 0$

$$H(x) = h_0 + h_1 x + h_2 x^2 + \cdots$$

$$-2xH(x) = \quad - 2h_0 x - 2h_1 x^2 - \cdots$$

$$-1/(1-x) = -1 - 1x - 1x^2 - \cdots$$

$$H(x) - 2xH(x) - 1/(1-x) = h_0 - 1 = -1$$

Hanoi Generating Function

$$H(x)(1-2x) = x/(1-x)$$

$$H(x) = \frac{x}{(1-x)(1-2x)}$$

(The gen func from last lecture)

Hanoi Generating Function

by partial fractions
from last lecture

$$[x^n]H(x) = 2^n - 1$$

linear recurrences

this method solves

$$f_n = a_1 f_{n-1} + a_2 f_{n-2} + \cdots + a_k f_{n-k} + 1$$

nonhomogeneous terms

handle	with
$\cdots + 1$	$1/(1-x)$
$\cdots + 2^n$	$1/(1-2x)$
$\cdots + n$	$x/(1-x)^2$
$\cdots + n^2$	$x(1+x)/(1-x)^3$
$\cdots + \alpha^n \cdot n^k$	$P(x)/Q(x)$

Team Problems

Problems 1 & 2