



# Generating Functions for Recurrences



## The Rabbit Population

- A *breeding pair* of rabbits produces a newborn pair every month.
- Rabbits breed after one month.
- After  $n$  months:

$w_n ::=$  # newborn pairs

$b_n ::=$  # breeding pairs

- Start with a newborn pair:  $w_0 = 1$   
 $b_0 = 0$



## The Rabbit Population

$w_n ::=$  # newborn pairs

$b_n ::=$  # breeding pairs

$$b_n = b_{n-1} + w_{n-1} \quad (\text{and so } b_1 = 1)$$

$$w_n = b_{n-1}$$

Therefore,

$$b_n = b_{n-1} + b_{n-2}$$



## The Rabbit Population

$$b_n = b_{n-1} + b_{n-2}$$

It was **Fibonacci** who was studying rabbit population growth!



## Generating Function for Rabbits

$$B(x) ::= b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$$

$$-xB(x) = -b_0x - b_1x^2 - b_2x^3 - \dots$$

$$-x^2B(x) = -b_0x^2 - b_1x^3 - \dots$$

$$\overset{\vee}{0}x^2 + \overset{\vee}{0}x^3 + \dots$$

$$b_n - b_{n-1} - b_{n-2} = 0$$



## Generating Function for Rabbits

$$\begin{array}{l} B(x) = b_0 + b_1x \\ -xB(x) = -b_0x \\ -x^2B(x) \end{array} \quad \left( \begin{array}{l} b_0 = 0 \\ b_1 = 1 \end{array} \right)$$

$$B(x) - xB(x) - x^2B(x) = b_0 + (b_1 - b_0)x$$

$$B(x)(1 - x - x^2) = 0 + (1 - 0)x = x$$





### Generating Function for Rabbits

$$B(x) = \frac{x}{1-x-x^2}$$



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### Coefficient notation

$[x^n]B(x) ::= b_n =$  coeff of  $x^n$  in power series for  $B(x)$

$$[x^n] \frac{1}{(1-\alpha x)} = \alpha^n$$



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### Generating Function for Rabbits

$$B(x) = \frac{x}{1-x-x^2}$$

use **partial fraction** expansion to find closed form for  $[x^n]B(x)$ .



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### Closed Form for $[x^n]B(x)$

$$B(x) = \frac{x}{1-x-x^2}$$

factor the denominator

$$1-x-x^2 = (1-\alpha x)(1-\beta x)$$

use **quadratic formula**



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### Closed Form for $[x^n]B(x)$

$$B(x) = \frac{x}{1-x-x^2}$$

factor the denominator

$$1-x-x^2 = (1-\alpha x)(1-\beta x)$$

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$$



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### Closed Form for $[x^n]B(x)$

$$B(x) = \frac{a}{1-\alpha x} + \frac{b}{1-\beta x}$$


$$[x^n]B(x) = a \cdot \alpha^n + b \cdot \beta^n$$

need to solve for **a** and **b**



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 **Closed Form for  $[x^n]B(x)$**


$$\frac{a}{1-\alpha x} + \frac{b}{1-\beta x} = \frac{x}{(1-\alpha x)(1-\beta x)}$$

Multiply both sides by  $(1-\alpha x)(1-\beta x)$

$$a(1-\beta x) + b(1-\alpha x) = x$$

Solve for  $a$  and  $b$  —letting  $x$  be  $1/\alpha$ , then  $1/\beta$  makes it easy


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 **the answer**

$$[x^n]B(x) = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

**amazing!**


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 **the answer**

$$[x^n]B(x) = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

$\approx 1.62$        $\approx -0.62$

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 **the answer**

$$[x^n]B(x) = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

$\approx 1.62$        $\approx (-0.62)^n$

converges to 0 as  $n$  grows


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 **the answer**

$$[x^n]B(x) \sim \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n \approx 0.45(1.62)^n$$

rabbit population grows exponentially


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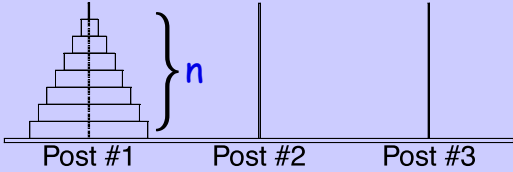
 **Simpler Closed Form**

$$[x^n]B(x) = \left\lfloor \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n \right\rfloor$$

round to nearest integer

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 Towers of Hanoi




Post #1    Post #2    Post #3

move stack to Post #2  
larger disc not above smaller

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 Towers of Hanoi



Post #1    Post #2    Post #3

$Move_{1,2}(n) ::= [Move_{1,3}(n-1);$   
big disk 1→2;  
 $Move_{3,2}(n-1)]$

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 Towers of Hanoi

$h_n ::= \# \text{ steps by } Move_{1,2}(n)$

$h_n = 2h_{n-1} + 1$

$h_0 = 0$

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 Hanoi Generating Function

$h_n = 2h_{n-1} + 1$

$h_0 = 0$

$H(x) = h_0 + h_1x + h_2x^2 + \dots$


$-2xH(x) = -2h_0x - 2h_1x^2 - \dots$

$-1/(1-x) = -1 - 1x - 1x^2 - \dots$

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$H(x) - 2xH(x) - 1/(1-x) = h_0 - 1 = -1$

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 Hanoi Generating Function

$H(x)(1-2x) = x/(1-x)$

$H(x) = \frac{x}{(1-x)(1-2x)}$

(The gen func from last lecture)

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 Hanoi Generating Function

by partial fractions  
from last lecture

$[x^n] H(x) = 2^n - 1$

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## linear recurrences

this method solves

$$f_n = a_1 f_{n-1} + a_2 f_{n-2} + \dots + a_k f_{n-k} + 1$$



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## nonhomogeneous terms handle with

$\dots + 1$	$1/(1-x)$
$\dots + 2^n$	$1/(1-2x)$
$\dots + n$	$x/(1-x)^2$
$\dots + n^2$	$x(1+x)/(1-x)^3$
$\dots + \alpha^n \cdot n^k$	$P(x)/Q(x)$



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## Team Problems

# Problems 1 & 2



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