Solutions to In-Class Problems Week 11, Fri.

Problem 1.
We are interested in generating functions for the number of different ways to compose a bag of n donuts subject to various restrictions. For each of the restrictions in (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.

Solution.

$$\langle 0,0,0,1,1,\ldots,1,\ldots\rangle \longleftrightarrow \frac{x^3}{1-x}$$

■

(b) All the donuts are glazed and there are at most 2.

Solution.

$$\langle 1,1,1,0,0,\ldots,0,\ldots\rangle \longleftrightarrow 1+x+x^2$$

■

(c) All the donuts are coconut and there are exactly 2 or there are none.

Solution.

$$\langle 1,0,1,0,0,\ldots,0,\ldots\rangle \longleftrightarrow 1+x^2$$

■

(d) All the donuts are plain and their number is a multiple of 4.

Solution.

$$\langle 1,0,0,0,1,0,0,0,\ldots,1,0,0,0,\ldots\rangle \longleftrightarrow \frac{1}{1-x^4}$$

■

(e) The donuts must be chocolate, glazed, coconut, or plain and:

- there must be at least 3 chocolate donuts, and
- there must be at most 2 glazed, and
- there must be exactly 0 or 2 coconut, and
- there must be a multiple of 4 plain.

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Solution.

$$\begin{aligned}\frac{x^3}{1-x}(1+x+x^2)(1+x^2)\frac{1}{1-x^4} &= \frac{x^3(1+x+x^2)(1+x^2)}{(1-x)^2(1+x)(1+x^2)}\\ &= x^3\frac{1+x+x^2}{(1-x)^2(1+x)}\end{aligned}$$

■

(f) Find a closed form for the number of ways to select n donuts subject to the constraints of the previous part.

Solution. Let

$$G(x) ::= \frac{1+x+x^2}{(1-x)^2(1+x)},$$

so the generating function for donut selections is $x^3G(x)$. By partial fractions

$$\frac{1+x+x^2}{(1-x)^2(1+x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} \tag{1}$$

for some constants, A, B, C. We know that the coefficient of x^n in the series for $(1-x)^2$ is, by the Convolution Rule, the number of ways to select n items of two different kinds, namely, $\binom{n+1}{1} = n+1$, so we conclude that the nth coefficient in the series for $G(x)$ is

$$A + B(n+1) + C(-1)^n. \tag{2}$$

To find A, B, C, we multiply both sides of (1) by the denominator $(1-x)^2(1+x)$ to obtain

$$1+x+x^2 = A(1-x)(1+x) + B(1+x) + C(1-x)^2. \tag{3}$$

Letting $x = 1$ in (3), we conclude that $3 = 2B$, so $B = 3/2$. Then, letting $x = -1$, we conclude $(-1)^2 = C2^2$, so $C = 1/4$. Finally, letting $x = 0$, we have

$$1 = A + B + C = A + \frac{3}{2} + \frac{1}{4},$$

so $A = -3/4$. Then from (2), we conclude that the nth coefficient in the series for $G(x)$ is

$$-\frac{3}{4} + \frac{3(n+1)}{2} + \frac{(-1)^n}{4} = \frac{6n+3+(-1)^n}{4}.$$

So the nth coefficient in the series for the generating function, $x^3G(x)$, for donut selections is zero for $n < 3$, and, for $n \geq 3$, is the $(n-3)$rd coefficient of G, namely,

$$\frac{6(n-3)+3+(-1)^{n-3}}{4} = \frac{6n-15+(-1)^{n-1}}{4}.$$

■

Problem 2. **(a)** Let

$$S(x) ::= \frac{x^2 + x}{(1-x)^3}.$$

What is the coefficient of x^n in the generating function series for $S(x)$?

Solution. n^2. That is, $S(x) = \sum_{n=1}^{\infty} n^2 x^n$.

To see why, note that the coefficient of x^n in $1/(1-x)^3$ is, by the Convolution Rule, the number of ways to select n items of three different kinds, namely,

$$\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}.$$

Now the coefficient of x^n in $x^2/(1-x)^3$ is the same as the coefficient of x^{n-2} in $1/(1-x)^3$, namely, $((n-2)+2)((n-2)+1)/2 = n(n-1)/2$. Similarly, the coefficient of x^n in $x/(1-x)^3$ is the same as the coefficient of x^{n-1} in $1/(1-x)^3$, namely, $((n-1)+2)((n-1)+1)/2 = (n+1)n/2$. The coefficient of x^n in $S(x)$ is the sum of these two coefficients, namely,

$$\frac{n(n-1)}{2} + \frac{(n+1)n}{2} = \frac{(n^2-n)+(n^2+n)}{2} = n^2.$$

■

(b) Explain why $S(x)/(1-x)$ is the generating function for the sums of squares. That is, the coefficient of x^n in the series for $S(x)/(1-x)$ is $\sum_{k=1}^{n} k^2$.

Solution.

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k \cdot 1\right) x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k\right) x^n \tag{4}$$

by the convolution formula for the product of series. For $S(x)$, the coefficient of x^k is $a_k = k^2$, and

$$S(x)/(1-x) = S(x)\left(\sum_{n=0}^{\infty} x^n\right),$$

so (4) implies that the coefficient of x^n in $S(x)/(1-x)$ is the sum of the first n squares. ■

(c) Use the previous parts to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. We have

$$\frac{S(x)}{1-x} = \frac{\left(\frac{x(1+x)}{(1-x)^3}\right)}{1-x} = \frac{x+x^2}{(1-x)^4}. \tag{5}$$

The coefficient of x^n in the series expansion of $1/(1-x)^4$ is

$$\binom{n+3}{3} = \frac{(n+1)(n+2)(n+3)}{3!}.$$

But by (5),

$$\frac{S(x)}{1-x} = \frac{x}{(1-x)^4} + \frac{x^2}{(1-x)^4},$$

so the coefficient of x^n is the sum of the $(n-1)$st and $(n-2)$nd coefficients of $(1-x)^4$, namely,

$$\frac{n(n+1)(n+2)}{3!} + \frac{(n-1)n(n+1)}{3!} = \frac{n(n+1)(2n+1)}{6}.$$

■

Appendix

Let $[x^n]F(x)$ denote the coefficient of x^n in the power series for $F(x)$. Then,

$$[x^n]\left(\frac{1}{(1-\alpha x)^k}\right) = \binom{n+k-1}{k-1}\alpha^n. \quad (6)$$

Partial Fractions

Here's a particular case of the Partial Fraction Rule that should be enough to illustrate the general Rule. Let

$$r(x) ::= \frac{p(x)}{(1-\alpha x)^2(1-\beta x)(1-\gamma x)^3}$$

where α, β, γ are distinct complex numbers, and $p(x)$ is a polynomial of degree less than the denominator, namely, less than 6. Then there are unique numbers $a_1, a_2, b, c_1, c_2, c_3 \in \mathbb{C}$ such that

$$r(x) = \frac{a_1}{1-\alpha x} + \frac{a_2}{(1-\alpha x)^2} + \frac{b}{1-\beta x} + \frac{c_1}{1-\gamma x} + \frac{c_2}{(1-\gamma x)^2} + \frac{c_3}{(1-\gamma x)^3}$$

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