Mathematics for Computer Science

MIT 6.042J/18.062J

Generating Functions

Infinite Geometric Sum

$$S(x) ::= 1 + x + x^2 + \cdots + x^n + \cdots$$
$$xS(x) = x + x^2 + \cdots + x^n + \cdots$$

Infinite Geometric Sum

$$S(x) ::= 1 + x + x^2 + \cdots + x^n + \cdots$$
$$xS(x) = x + x^2 + \cdots + x^n + \cdots$$

$$S(x) - xS(x) = 1$$

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

Ordinary Generating Functions

The ordinary generating function for the infinite sequence

$$\langle g_0, g_1, g_2, \cdots, g_n, \cdots \rangle$$

is the power series:

$$G(x) = g_0 + g_1 x + g_2 x^2 + \cdots + g_n x^n + \cdots$$

Infinite Geometric Sum

"corresponds to"

$$\langle 1, 1, 1, \ldots \rangle \longleftrightarrow$$

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

Bags of fruit

In how many ways can we fill a bag with n fruits given the following constraints?

- At most 2 oranges.
- Any number of apples.
- Any number of bananas that only come in bunches of 3.

Bags with n = 4 fruits

- 0 oranges, 1 apple, 3 bananas
- 0 oranges, 4 apples, 0 bananas
- 1 orange, 0 apples, 3 bananas
- 1 orange, 3 apples, 0 bananas
- 2 oranges, 2 apples, 0 bananas

Number of 4-fruit bags: 5

At most 2 oranges

$$O(x) = \sum_{k=0}^{\infty} o_k x^k$$

ways to pick k oranges

$o_0 = 1$

$o_1 = 1$

$o_2 = 1$

$o_k = 0 \quad k \geq 3$

$$O(x) = 1 + x + x^2 = \frac{1-x^3}{1-x}$$

Any number of apples

There is only 1 way to pick a bag of k apples: $a_k = 1$

$$A(x) = a_0 1 + a_1 x + a_2 x^2 + \cdots$$

$$= 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

Substituting x^k for x

$$\frac{1}{1-x} \longleftrightarrow \langle 1,1,1,1,\ldots \rangle$$

$$\frac{1}{1-x^k} = 1 + x^k + x^{2k} + x^{3k} \cdots$$

$$\longleftrightarrow \langle 1, \underbrace{0,\ldots,0}_{k-1 \text{ zeros}}, 1, \underbrace{0,\ldots,0}_{k-1 \text{ zeros}}, 1, 0, \ldots \rangle$$

Bananas in bunches of 3

$$B(x) \longleftrightarrow \langle 1,0,0,1,0,0,1,0,\ldots \rangle$$

$$B(x) = \frac{1}{1-x^3}$$

Convolution Rule

$$O(x) = \frac{1-x^3}{1-x}$$

$$A(x) = \frac{1}{1-x}$$

$$B(x) = \frac{1}{1-x^3}$$

We can use the individual generating functions to solve original fruit problem!

Convolution Rule

Ways to pick 12 apples & bananas:

	# ways
• 0 apples and 12 bananas	1
• 1 apple and 11 bananas	0
⋮	⋮
• 12 apples and 0 bananas	1

Total=5

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Convolution Rule

Ways to pick 12 apples & bananas:

a_j = # ways to pick j apples

b_k = # ways to pick k bananas

$a_j b_{12-j}$ = # ways to pick j apples *and* rest bananas

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Convolution Rule

ways to pick 12 apples & bananas

$$= a_0 b_{12} + a_1 b_{11} + \ldots + a_{11} b_1 + a_{12} b_0$$

But this is the coefficient of x^{12} in $A(x) \cdot B(x)$

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Convolution Rule

The coefficient of x^{12} in the product $A(x) \cdot B(x)$:

$$(a_0 x^0 + a_1 x^1 + \cdots + a_{11} x^{11} + a_{12} x^{12} + \cdots) \times$$
$$(b_0 x^0 + b_1 x^1 + \cdots + b_{11} x^{11} + b_{12} x^{12} + \cdots)$$

$$a_0 b_{12} + a_1 b_{11} + \ldots + a_{11} b_1 + a_{12} b_0$$

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Convolution Rule

ways to pick 12 apples & bananas

is the coefficient of x^{12} in

$$\underbrace{A(x) \cdot B(x)}$$

the generating function for picking apples & bananas

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Convolution Rule

The gen func for choosing from a union of disjoint sets is the *product* of the gen funcs for choosing from each set.

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Bags of Fruit

Gen func for the bags of fruit:

$$F(x) = O(x)\cdot A(x)\cdot B(x)$$

$$= \frac{\cancel{1-x^3}}{1-x}\cdot\frac{1}{1-x}\cdot\frac{1}{\cancel{1-x^3}}$$

$$= \frac{1}{(1-x)^2}$$

A Familiar Generating Function?

so # of our bags with n fruits is the coefficient of x^n in

$$1/(1-x)^2$$

We can easily relate $1/(1-x)^2$ to something we already know how to count!

A Familiar Generating Function?

The gen func for selecting n donuts of a given flavor:

$$D_{chocolate} = \frac{1}{1-x}$$

$$D_{vanilla} = \frac{1}{1-x}$$

A Familiar Generating Function?

The gen func for selecting n donuts using both flavors:

$$D_{chocolate}\cdot D_{vanilla} = \left(\frac{1}{1-x}\right)\cdot\left(\frac{1}{1-x}\right)$$

$$= \frac{1}{(1-x)^2}$$

A Familiar Generating Function?

The gen func for selecting n donuts among k flavors:

$$= \frac{1}{(1-x)^k}$$

The Donut Number!

Using k different flavors, how many ways are there to form a bag of n donuts?

(You already know the answer to this one.)

The Donut Number!

Using k different flavors, how many ways are there to form a bag of n donuts?

$$\binom{n+k-1}{n}$$

The Donut Number!

so coeff of x^n in $\frac{1}{(1-x)^k}$ is

$$\binom{n+k-1}{n}$$

Conclusion: Bags of Fruit

In how many ways can we fill a bag with n of our fruits?

$$F(x) = \frac{1}{(1-x)^2}$$

$$f_n = \binom{n+2-1}{n} = n+1$$

Finding coefficients

If a generating function $H(x)$ is a *quotient of polynomials* there is a simple way to find the nth coefficient h_n

Partial Fraction Expansions

$$H(x) ::= \frac{x}{2x^2-3x+1}$$

Factor denominator

$$= \frac{x}{(1-2x)(1-x)}$$

Express as sum

$$= \frac{A_1}{1-2x} + \frac{A_2}{1-x}$$

Partial Fractions for $H(x)$

$$H(x) = A_1\left[\frac{1}{1-2x}\right] + A_2\left[\frac{1}{1-x}\right]$$

$$h_n = A_1 \cdot 2^n + A_2 \cdot 1$$

TO DO: find A_1 and A_2.

Solve for A_1 and A_2

$$\frac{x}{(1-2x)(1-x)} = \frac{A_1}{1-2x} + \frac{A_2}{1-x}$$

Multiply both sides by denom of LHS.

$$x = A_1(1-x) + A_2(1-2x)$$

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Solve for A_1 and A_2

$$x = A_1(1-x) + A_2(1-2x)$$

Substitute in values for x.

$x = 1$:

$1 = A_2(1-2)$ $\boxed{A_2 = -1}$

$x = 1/2$:

$1/2 = A_1(1-1/2)$ $\boxed{A_1 = 1}$

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Finding the coefficients

$$H(x) = \frac{1}{1-2x} - \frac{1}{1-x}$$

the partial fraction expansion

$$h_n = 2^n - 1$$

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In General…

The partial fraction expansion of $P(x)/Q(x)$ contains terms of the form

$$\cdots + \frac{A}{(1-\alpha x)^k} + \cdots$$

We know the n^{th} coeff of this!

$$A \cdot \binom{n+k-1}{n} \alpha^n$$

where $1/\alpha$ is a root of $Q(x)$.

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Partial Fractions Caveat #1

For roots with multiplicity $k>1$ in factored denominator of gen func

$$\frac{\cdots}{(1-\alpha x)^k \cdots}$$

need k partial fractions:

$$\frac{A_1}{(1-\alpha x)^1} + \frac{A_2}{(1-\alpha x)^2} + \cdots + \frac{A_k}{(1-\alpha x)^k} + \cdots$$

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Partial Fractions Caveat #2

Gen func $F(x) = N(x)/D(x)$

If $\deg(N) > \deg(D)$…

use polynomial long division to find $Q(x)$ and $R(x)$ such that

$$F(x) = Q(x) + R(x)/D(x)$$

and $\deg(R) < \deg(D)$.

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Team Problems

Problems 1 & 2

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