Solutions to In-Class Problems Week 11, Wed.

Problem 1.
Find the coefficients of

(a) x^5 in $(1+x)^{11}$

Solution.

$$\binom{11}{5} = 462$$

■

(b) x^8y^9 in $(3x+2y)^{17}$

Solution.

$$\binom{17}{8}3^8 2^9.$$

When $(3x+2y)^{17}$ is expressed as a sum of powers of the summands $3x$ and $2y$, the coefficient of $(3x)^8(2y)^9$ is $\binom{17}{8}$, so the coefficient of x^8y^9 is this binomial coefficient times $3^8 \cdot 2^9$. ■

(c) a^6b^6 in $(a^2+b^3)^5$

Solution. $a^6b^6 = (a^2)^3(b^3)^2$, so the coefficient is

$$\binom{5}{3} = 10$$

■

Problem 2.
You want to choose a team of m people for your startup company from a pool of n applicants, and from these m people you want to choose k to be the team managers. You took 6.042, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Before doing the reasonable thing —dump on your CFO or Harvard Business School —you decide to check his answer against yours.

Creative Commons 2010, Prof. Albert R. Meyer.

(a) Give a *combinatorial proof* that your CFO's formula agrees with yours.

Solution. Instead of choosing first m from n and then k from the chosen m, you could alternately choose the k managers from the n people and then choose $m - k$ people to fill out the team from the remaining $n - k$ people. This gives you $\binom{n}{k}\binom{n-k}{m-k}$ ways of picking your team. Since you must have the same number of options regardless of the order in which you choose to pick team members and managers,

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

Formally, in the first method we count the number of pairs (A, B), where A is a size m subset of the pool of n applicants, and B is a size k subset of A. By the Generalized Product Rule, there are

$$\binom{n}{m} \cdot \binom{m}{k}$$

such pairs.

In the second method, we count pairs (C, D), where C is a size k subset of the applicant pool, and D is a size $(m - k)$ subset of the pool that is disjoint from C. By the Generalized Product Rule, there are

$$\binom{n}{k} \cdot \binom{n-k}{m-k}$$

such pairs.

These two expressions are equal because there is an obvious bijection between the two kinds of pairs, namely map (A, B) to $(B, A - B)$. ■

(b) Verify this combinatorial proof by giving an *algebraic* proof of this same fact.

Solution.

$$\begin{aligned}
\binom{n}{m}\binom{m}{k} &= \frac{n!}{m!(n-m)!}\frac{m!}{k!(m-k)!} \\
&= \frac{n!}{(n-m)!k!(m-k)!} \\
&= \frac{n!(n-k)!}{(n-m)!k!(m-k)!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(n-m)!(m-k)!} \\
&= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{((n-k)-(m-k))!(m-k)!} \\
&= \binom{n}{k}\binom{n-k}{m-k}.
\end{aligned}$$

■

Problem 3. **(a)** Now give a combinatorial proof of the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} \tag{1}$$

Hint: Let S be the set of all length-n sequences of 0's, 1's and a single *.

Solution. Let $P ::= \{0, \ldots, n-1\} \times \{0,1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the kth bit in the length-$n-1$ binary word, x. So

$$|S| = |P| = n2^{n-1} \tag{2}$$

by the Product Rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the $*$, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the * among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:

$$|S| = \sum_{k=1}^{n} k\binom{n}{k}$$

Equating this expression and the expression (2) for $|S|$ proves the theorem. ■

(b) Now prove (1) algebraically by applying the Binomial Theorem to $(1+x)^n$ and taking derivatives.

Solution. By the Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Taking derivatives, we get

$$\begin{aligned} n(1+x)^{n-1} &= \sum_{k=0}^{n} k\binom{n}{k} x^{k-1} \\ &= \frac{1}{x}\sum_{k=0}^{n} k\binom{n}{k} x^k. \end{aligned} \tag{3}$$

Letting $x = 1$ in (3) yields (1).

■

MIT OpenCourseWare
http://ocw.mit.edu

6.042J / 18.062J Mathematics for Computer Science
Spring 2010

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.