



# Binomial Theorem, Combinatorial Proof



## Polynomials Express Choices & Outcomes

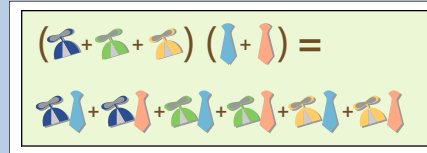


Image by MIT OpenCourseWare.

**Products of Sums = Sums of Products**



## expression for $c_k$ ?

$$(1+X)^n =$$

$$c_0 + c_1X + c_2X^2 + \dots + c_nX^n$$



## expression for $c_k$ ?

$$(1+X)^n \quad n \text{ times}$$

$$= (1+X)(1+X)(1+X)(1+X)\dots(1+X)$$

multiplying gives  $2^n$  product terms:

$$11\dots 1 + X11X\dots X1 + 1XX\dots X1 + \dots + XX\dots X$$

a term corresponds to selecting 1 or X from each of the  $n$  factors



## expression for $c_k$ ?

$$(1+X)^n \quad n \text{ times}$$

$$= (1+X)(1+X)(1+X)(1+X)\dots(1+X)$$

the  $X^k$  coeff,  $c_k$ , is # terms with exactly  $k$  X's selected

$$c_k = \binom{n}{k}$$



## The Binomial Formula


binomial expression

$$(1+X)^n =$$

$$\binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \dots + \binom{n}{k}X^k + \dots + \binom{n}{n}X^n$$

binomial coefficients





 **The Binomial Formula**

$$(X + Y)^n =$$


$$\binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} +$$


$$\dots + \binom{n}{k} x^k y^{n-k} + \dots + \binom{n}{n} x^n$$

 Albert R Meyer, April 21, 2010 lec.11W.8

 **The Binomial Formula**


$$(X + Y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$


 Albert R Meyer, April 21, 2010 lec.11W.9

 **multinomial coefficients**

What is the coefficient of  $x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_k^{r_k}$  in the expansion of  $(x_1 + x_2 + x_3 + \dots + x_k)^n$  ?


$$\binom{n}{r_1, r_2, r_3, \dots, r_k}$$


 Albert R Meyer, April 21, 2010 lec.11W.17

 **The Multinomial Formula**

$$(x_1 + x_2 + \dots + x_k)^n =$$


$$\sum_{r_1 + \dots + r_k = n} \binom{n}{r_1, r_2, \dots, r_k} x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_k^{r_k}$$


 Albert R Meyer, April 21, 2010 lec.11W.18

 **multinomial coefficients**

binomial a special case:

$$\binom{n}{k} = \binom{n}{k, n-k}$$

 Albert R Meyer, April 21, 2010

 **More next lecture about counting with polynomials and series**





Image by MIT OpenCourseWare.

 Albert R Meyer, April 21, 2010 lec.11W.20

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Preceding slides adapted from:

- *Great Theoretical Ideas In Computer Science*  
Carnegie Mellon Univ., CS 15-251, Spring 2004  
Lecture 10 Feb 12, 2004 by Steven Rudich
- *Applied Combinatorics*, by Alan Tucker



Albert R Meyer, April 21, 2010

lec 11W.21

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

**Algebraic Proof** : routine, using

$$\binom{n}{k} ::= \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k} \binom{n-1}{k-1}$$



Albert R Meyer, April 21, 2010

lec 11W.22

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

# size k subsets =  
# size k subsets **with 1**  
+ # size k subsets **without 1**



Albert R Meyer, April 21, 2010

lec 11W.23

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} = \binom{n-1}{k} + \binom{n-1}{k-1}$$



Albert R Meyer, April 21, 2010

lec 11W.24

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} = \underbrace{\binom{n-1}{k}}_{\text{\# size k subsets without 1}} + \binom{n-1}{k-1}$$



Albert R Meyer, April 21, 2010

lec 11W.25

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Combinatorial Proof

classify subsets of  $\{1, \dots, n\}$

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} \stackrel{\text{QED}}{=} \underbrace{\binom{n-1}{k}}_{\text{\# size k subsets without 1}} + \underbrace{\binom{n-1}{k-1}}_{\text{\# size k subsets with 1}}$$



Albert R Meyer, April 21, 2010

lec 11W.26



## Combinatorial Proof, II

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$



Albert R Meyer,

April 21, 2010

lec.11W.27



## Combinatorial Proof, II

classify subsets of  $\{1, \dots, n, 1, \dots, n\}$

$$\text{RHS} = \underbrace{\binom{2n}{n}}_{\substack{\# \text{ size } n \\ \text{subsets}}}$$



Albert R Meyer,

April 21, 2010

lec.11W.28



## Combinatorial Proof, II

$$\begin{aligned} \text{LHS} &= \sum_{i=0}^n \binom{n}{i}^2 \\ &= \sum_{i=0}^n \binom{n}{i} \binom{n}{n-i} \end{aligned}$$



Albert R Meyer,

April 21, 2010

lec.11W.29



## Combinatorial Proof, II

LHS =

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\text{size } i} \binom{n}{n-i}$$



Albert R Meyer,

April 21, 2010

lec.11W.30



## Combinatorial Proof, II

LHS =

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\# \text{ size } i \\ \text{red subsets}}} \binom{n}{n-i}$$



Albert R Meyer,

April 21, 2010

lec.11W.30



## Combinatorial Proof, II

LHS =

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\# \text{ size } i \\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\# \text{ size } n-i \\ \text{black subsets}}}$$



Albert R Meyer,

April 21, 2010

lec.11W.31



## Combinatorial Proof, II

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\# \text{ size } i \\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\# \text{ size } n-i \\ \text{black subsets}}}$$

So LHS = # size n subsets  
of  $\{1, \dots, n, 1, \dots, n\}$  by Sum Rule



Albert R Meyer,

April 21, 2010

lec 11W.32



## Combinatorial Proof, II

Therefore

LHS = # size n subsets = RHS

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

QED



Albert R Meyer,

April 21, 2010

lec 11W.33



## Team Problems

# Problems 1-3



Albert R Meyer,

April 21, 2010

lec 11W.34

MIT OpenCourseWare  
<http://ocw.mit.edu>

6.042J / 18.062J Mathematics for Computer Science  
Spring 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.