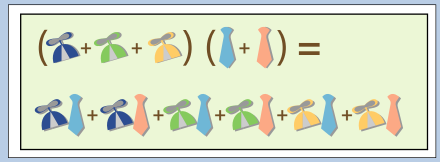
Mathematics for Computer Science

MIT 6.042J/18.062J

Binomial Theorem, Combinatorial Proof

Polynomials Express Choices & Outcomes



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Products of Sums = Sums of Products

expression for c_k?

$$(1+X)^n =$$

$$c_0 + c_1 X + c_2 X^2 + \ldots + c_n X^n$$

expression for c_k?

$(1+X)^n$ n times

$= (1+X)(1+X)(1+X)(1+X)\ldots(1+X)$

multiplying gives 2^n product terms:

$11\cdots1 + X11X\cdots X1 + 1XX\cdots1X1 + \cdots + XX\cdots X$

a term corresponds to selecting 1 or X from each of the n factors

expression for c_k?

$(1+X)^n$ n times

$= (1+X)(1+X)(1+X)(1+X)\ldots(1+X)$

the X^k coeff, c_k, is # terms with exactly k X's selected

$$c_k = \binom{n}{k}$$

The Binomial Formula

binomial expression

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

binomial coefficients

The Binomial Formula

$$(X+Y)^n =$$

$$\binom{n}{0}Y^n + \binom{n}{1}XY^{n-1} + \binom{n}{2}X^2Y^{n-2} + \ldots + \binom{n}{k}X^kY^{n-k} + \ldots + \binom{n}{n}X^n$$

The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{n}\binom{n}{k}X^kY^{n-k}$$

multinomial coefficients

What is the coefficient of

$$X_1^{r_1}X_2^{r_2}X_3^{r_3}\ldots X_k^{r_k}$$

in the expansion of

$$(X_1+X_2+X_3+\ldots+X_k)^n \text{ ?}$$

$$\binom{n}{r_1,r_2,r_3,\ldots,r_k}$$

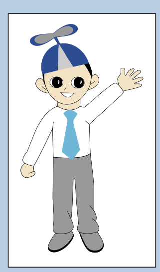
The Multinomial Formula

$$\left(X_1+X_2+\ldots+X_k\right)^n = \sum_{r_1+\cdots+r_k=n}\binom{n}{r_1,r_2,\ldots,r_k}X_1^{r_1}X_2^{r_2}X_3^{r_3}\ldots X_k^{r_k}$$

multinomial coefficients

binomial a special case:

$$\binom{n}{k} = \binom{n}{k,\ n-k}$$



More next lecture about counting with polynomials and series

Image by MIT OpenCourseWare.

Preceding slides adapted from:

- *Great Theoretical Ideas In Computer Science*
  Carnegie Mellon Univ., CS 15-251, Spring 2004
  Lecture 10 Feb 12, 2004 by Steven Rudich
- *Applied Combinatorics*, by Alan Tucker

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Algebraic Proof : routine, using

$$\binom{n}{k} ::= \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of {1,…,n}

\# size k subsets =
\# size k subsets with 1
\+ # size k subsets without 1

Combinatorial Proof

classify subsets of {1,…,n}

$$\underbrace{\binom{n}{k}}_{\#\text{ size } k \text{ subsets}} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of {1,…,n}

$$\underbrace{\binom{n}{k}}_{\#\text{ size } k \text{ subsets}} = \underbrace{\binom{n-1}{k}}_{\#\text{ size } k \text{ subsets without } 1} + \binom{n-1}{k-1}$$

Combinatorial Proof

classify subsets of {1,…,n}

QED

$$\underbrace{\binom{n}{k}}_{\#\text{ size } k \text{ subsets}} = \underbrace{\binom{n-1}{k}}_{\#\text{ size } k \text{ subsets without } 1} + \underbrace{\binom{n-1}{k-1}}_{\#\text{ size } k \text{ subsets with } 1}$$

Combinatorial Proof, II

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Combinatorial Proof, II

classify subsets of {1,….,n,1,….,n}

$$\text{RHS} = \underbrace{\binom{2n}{n}}_{\text{\# size n subsets}}$$

Combinatorial Proof, II

$$\text{LHS} = \sum_{i=0}^{n} \binom{n}{i}^2$$

$$= \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}} \binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\text{\# size i red subsets}} \binom{n}{n-i}$$

Combinatorial Proof, II

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\text{\# size i red subsets}} \underbrace{\binom{n}{n-i}}_{\text{\# size n-i black subsets}}$$

Combinatorial Proof, II

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\substack{\#\text{ size } i \\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\#\text{ size } n-i \\ \text{black subsets}}}$$

So LHS = # size n subsets of {1,…,n,1,….,n} by Sum Rule

Combinatorial Proof, II

Therefore

LHS = # size n subsets = RHS

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

QED

Team Problems

Problems 1–3